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ELEMENTS

OF

ALGEBRA,

BEING AN

ABRIDGMENT OF DAY'S ALGEBRA,

ADAPTED TO THE

CAPACITIES OF THE YOUNG,

AND THE

METHOD OF INSTRUCTION,

IN

SCHOOLS AND ACADEMIES.

BY

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N. B. The Key to this work will shortly be published for the use of teachers.

An abridgment of LEGENDRE's Geometry, by the same author, will also be published for the use of Schools and Academies.

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PREFACE.

PUBLIC opinion has pronounced the study of Algebra to be a desirable and important branch of popular education. This decision is one of the clearest proofs of an onward and substantial progress in the cause of intellectual improvement in our country. A knowledge of algebra may not indeed be regarded as *strictly necessary* to the discharge of the common duties of life; nevertheless no young person at the present day is considered as having a "finished education" without an acquaintance with its rudiments.

The question with parents is, not "*how little learning* and *discipline* their children can get through the world with;" but, "*how much does their highest usefulness require;*" and "what are the *best means* to secure this end?"

It has long been a prevalent sentiment among teachers and the friends of education, that an abridgment of Day's Algebra, adapted to the wants of schools and academies, would greatly facilitate this object. Whilst his system has been deemed superior to any other work before the public, and most happily adapted to the circumstances of college students, for whom it was especially prepared; it has also been felt, that a smaller and cheaper work, combining the *simplicity* of language and the *unrivalled clearness* with which the principles of the science are there stated, would answer every purpose for beginners, and at the same time bring the subject within the means of the humblest child in the land.

In accordance with this sentiment, such a work has been prepared, and is now presented to the public. The design of the work is to furnish an *easy* and *lucid transition* from the study of arithmetic to the higher branches of algebra and mathematics, and thus to subserve the important interests of a practical and thorough education.

Its arrangement, with but few exceptions, is the same as that of the large work. For the sake of more convenient reference, the division by compound divisors, and the binomial theorem, both of which were originally placed after mathematical infinity, are brought forward, the former being placed after division of simple quantities, and the latter after involution of simple quantities. The reason for deferring the consideration of compound division in the original, was the fact that some of the terms contain powers which it is impossible for pupils at this stage of their progress to understand. To avoid this difficulty in the present work, whenever a power occurs, instead of using an index before it has been explained, the letter is repeated as a factor in the same manner as in multiplication, and also in dividing by a simple quantity. (Arts. 80, 94.) Afterwards, under division of powers, copious examples of dividing by compound quantities which have indices, are given.

As continued arithmetical proportion and arithmetical progression are one and the same thing, they are placed contiguously in the same section. For the same reason continued geometrical proportion and geometrical progression are placed in a similar manner. Mathematical infinity, roots of binomial surds, infinite series, indeterminate co-efficients, composition and resolution of the higher equations, with equations of curves, are subjects which belong to the higher and more difficult parts of algebra, and it has been thought advisable to omit them in the abridgment. Those who have

leisure and are desirous of acquiring a knowledge of these subjects, will find them explained with all the author's accustomed clearness and ability in his large work, to which they are respectfully referred. The similarity between the operations in addition, subtraction, multiplication and division of radical quantities, and those of the same rules in powers; also between involution and evolution of radicals, and of powers, has been more fully developed, and the rules of both are expressed in as nearly the same language as the nature of the case would admit. It has also been attempted to illustrate the "Binomial Theorem," on the principles of induction; the second method of completing the square in quadratic equations has been demonstrated; and other methods of completing the square pointed out, which, so far as the author knows, are original.

It was a cardinal point with the distinguished author of the large work, never to use one principle in the explanation of another, until it had itself been explained, a characteristic of rare excellence in school-books and works of science. This plan has been rigidly adhered to, in the preparation of the abridgment. After the principles have been separately explained, and illustrated by examples, they have then been carefully summed up in the present work, and placed in the form of a general rule. This, it is thought by competent judges, will be found very convenient and useful both to teachers and scholars. By this means the peculiar advantages of the inductive and synthetic modes of reasoning have been united, and made subservient alike to the pleasure and facility both of imparting and acquiring knowledge.

As a guide to the attention of beginners to the more important principles of the science, a few practical questions are placed at the foot of the page. They are intended to be merely *suggestive.* No thorough teacher will *confine* himself

to the questions of an author, however full and appropriate they may be. From a conviction that the answers to problems have a tendency to *destroy* rather than *promote* habits of independent thinking and reasoning in the minds of learners, they have nearly all been excluded from the book. For the convenience of teachers and others, who may entertain different views upon this point, the answers are given in a Key, in which may also be found a statement and solution of the more difficult examples contained in the work.

The formation of *correct* habits of study and of thought, together with the *extermination* or *prevention* of *bad* ones, requires the utmost vigilance and skill on the part of teachers. They must insist upon *thoroughness*, upon "the *why* and *wherefore*" of each successive step, or, in most cases, their pupils will fall into *superficial* and *mechanical habits*, which are equally destructive of high attainments and future usefulness. To mould the youthful mind *right*, is an *arduous* and *responsible* task; sufficient to crush the jaded spirit and shattered nerves of a poorly paid teacher. Nevertheless it is a *high* and *noble*, as well as *indispensable* work. Every conscientious teacher therefore, who appreciates the importance of his profession, or is worthy to be entrusted with this responsible charge, will cheerfully devote his energies to the work, whatever may be the sacrifice, or resign his trust to more faithful and able hands. "In mathematics as in war, it should be made a principle," says the author of the large work, "not to *advance*, while anything is left *unconquered* behind. Neither is it sufficient that the student understands the *nature* of the proposition, or method of operation, before proceeding to another. He ought also to make himself *familiar* with every step, by a careful attention to the examples." It is emphatically true in algebra, that "practice makes perfect." For this reason the number of problems in the present work has been nearly

doubled; the most of those added are original, and are calculated to make the principles of the science more familiar.

The merits of DAY'S ALGEBRA are too well *known* and *appreciated* to require any comment. The fact that it has been adopted, as a text-book, by so many of our colleges and higher seminaries of learning; that during the last fourteen years more than forty large editions have been called for, affords sufficient evidence of the superior rank which it holds in public estimation.

With regard to the abridgment, it is fervently hoped that all who have felt the want of a lucid introductory work upon this subject, will here find the fulfilment of their wishes. Those teachers who have used the large work in their collegiate course or elsewhere, and who may have occasion to use this, will at least be saved from the inconvenience of *unlearning* one set of rules, and of *learning* a new and perhaps an inferior set, a work by no means *unfrequent*, and of no small *magnitude* and *perplexity*. On the other hand, those scholars, who chance to use the abridgment in their preparatory course, will avoid the necessity of *unlearning* its rules and modes of operation in algebra, should they have occasion to use the large work in the subsequent part of their education.

It has been the endeavor of the author to divest the study of algebra, once so formidable, of all its intricacy and repulsiveness; to illustrate its elementary principles so *clearly*, that any school-boy of ordinary capacity may understand and apply them; and thus to render this interesting and useful science more attractive to the young. With what success these efforts have been attended, it remains for his fellow teachers and an impartial public to decide.

J. B. THOMSON.

New Haven, May 20, 1843.

NOTICE.

Having been myself prevented, by impaired health, and official engagements, from preparing an abridgment of my *Introduction to Algebra*, I applied to Mr. J. B. Thomson, to abridge the work, in such a manner as to adapt it to the demand and use of the higher schools and academies.

I had confidence in his mathematical talents and attainments, and his practical knowledge, derived from several years' experience in teaching algebra, as qualifying him to make the abridgment proposed; and I am gratified to find, on examination, that our design has been skilfully and satisfactorily executed. The abridgment, it is hoped, will be favorably received, by those who approve of the original work

J. DAY.

Yale College, May 29, 1843.

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ALGEBRA.

SECTION I.

INTRODUCTION.

ART. 1. ALGEBRA *is a general method of solving problems, and of investigating the relations of quantities by means of letters and signs.*

ILLUSTRATION.

PROB. 1. Suppose a man divided 72 dollars among his three sons in the following manner: To A he gave a certain number of dollars; To B he gave three times as many as to A; and to C he gave the remainder, which was half as many dollars as A and B received. How many dollars did he give to each?

1. To solve this problem *arithmetically*, the pupil would reason thus: A had a *certain part*, i. e. *one share;* B received *three* times as much, or *three shares;* but C had *half* as much as A and B; hence he must have received *two shares.* By adding their respective shares, the sum is *six shares*, which by the conditions of the question is equal to 72 dols. If then 6 shares are equal to 72 dols., 1 share is equal to $\frac{1}{6}$ of 72, viz. 12 dols., which is A's share. B had *three* times as many, viz. 36 dols., and C *half* as many dols. as both, viz. 24 dols.

QUEST.—What is algebra? How solve Prob. 1 arithmetically?

2. Now to solve the same problem by *algebra*, he would use *letters* and *signs;* thus,

Let x represent A's share; then by the conditions,

$x\times 3$ will represent B's share; and

$4x\div 2$ will represent C's share.

Add together the *several shares*, or x s; thus, $x+3x+2x=6x$. Then will $6x=72$, for the whole is equal to all its parts; and $1x=12$ dols. A's share; $3x=36$ dols. B's share; and $2x=24$ dols. C's share.

PROOF. Add together the number of dollars received by each, and the sum will be equal to 72, the amount divided.

In this *algebraic solution* it will be observed; *First*, that we *represent* the *number* of *dollars* which A received by x. *Second*, to obtain B's share, we must *multiply* A's share by 3. This *multiplication* is represented by *two lines crossing each other like a capital* X. *Third*, to find C's share, we must take *half* the sum of A's and B's share. This *division* is denoted by a *line between two dots*. *Fourth*, the *addition* of their respective shares is denoted by *another cross formed by a horizontal* and *perpendicular line*. Take another example.

PROB. 2. A boy wishes to lay out 96 cents for peaches and oranges, and wants to get an equal number of each. He finds that he must give 2 cents for a peach and 4 for an orange. How many can he buy of each?

Let x denote the number of each. Now since the price of one peach is 2 cents, the price of x peaches will be $x\times 2$ cents, or $2x$ cents. For the same reason $x\times 4$, or $4x$ cents, will denote the price of x oranges. Then will $2x+4x$, that is, $6x$, be equal to 96 cents by the conditions, and $1x$ is equal

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QUEST.—How by algebra? How denote A's share? How B's and C's? What is the share of each? In Prob. 2, how represent the number of each kind? What represents the price of each kind? The Ans.?

to $\frac{1}{6}$ of 96 cents, viz. 16 cents, which is the number he bought of each.

2. QUANTITIES in algebra are generally expressed by *letters*, as in the preceding problems. Thus b may be put for 2 or 15, or any other number which we may wish to express. It must not be inferred, however, that the letter used, has *no determinate* value. Its value is *fixed* for the *occasion* or *problem* on which it is employed; and remains *unaltered* throughout the solution of that problem. But on a *different* occasion, or in *another* problem, the same letter may be put for any other number. Thus in Prob. 1, x was put for A's share of the money. Its value was 12 dols. and remained fixed through the operation. In Prob. 2, x was put for the number of each kind of fruit. Its value was 16, and it remained so through the calculation.

3. By the term *quantity*, we mean anything which can be *multiplied*, *divided*, or *measured*. Thus a *line*, *weight*, *time*, *number*, &c., are called quantities.

4. The *first* letters of the alphabet are used to express *known* quantities; and the *last* letters, those which are *unknown*.

5. *Known* quantities are those whose values are given, or may be easily inferred from the conditions of the problem under consideration.

6. *Unknown* quantities are those whose values are not given.

7. *Sometimes*, however, the quantities, instead of being expressed by letters, are set down in *figures*.

QUEST.—How are quantities expressed in algebra? What does each letter stand for? Has the letter used no determinate value? What is meant by quantity? Give examples. Which letters are used to denote known quantities? Which unknown? What are known quantities? Unknown? Are figures ever used in algebra?

8. Besides letters and figures, it will also be seen that we use *certain signs* or *characters* in algebra to indicate the *relations* of the quantities, or the *operations* which are to be performed with them, instead of writing out these relations and operations in words. Among these is the sign of addition ($+$), subtraction ($-$), equality ($=$), &c.

9. *Addition* is represented by two lines ($+$), one horizontal, the other perpendicular, forming a cross, and is called *plus*. It signifies "more," or "added to." Thus $a+b$ signifies that b is to be added to a. It is read a plus b, or a added to b, or a and b.

10. *Subtraction* is represented by a short horizontal line ($-$) which is called *minus*. Thus $a-b$, signifies that b is to be "subtracted" from a; and is read a minus b, or a less b.

11. The sign $+$ is prefixed to quantities which are considered as *positive* or *affirmative;* and the sign $-$, to those which are supposed to be *negative*. For the nature of this distinction, see Arts. 36 and 37.

12. The sign is generally omitted before the *first* or *leading* quantity, unless it is *negative ;* then it must always be written. When *no sign* is prefixed to a quantity, $+$ is *always* understood. Thus $a+b$ is the same as $+a+b$.

13. Sometimes *both* $+$ and $-$ (the latter being put under the former, $\pm$) are prefixed to the *same* letter. The sign is then said to be *ambiguous*. Thus $a \pm b$ signifies, that in cer-

QUEST.—What besides letters and figures are used in algebra? What is the sign of addition? How read? What does it signify? How is subtraction represented? What called? What signify? What sign have positive quantities? What negative? What is said as to the sign of the leading quantity? When none is expressed what sign is understood? When both $+$ and $-$ are prefixed to the same letter, what is the sign called? What does it show? What are like signs? What unlike?

tain cases, comprehended in a general solution, b is to be add ed to a, and in other cases subtracted from it.

Obser. When *all* the signs are *plus*, or *all minus*, they are said to be *alike*; when *some* are plus and *others* minus, they are called *unlike*.

14. The *equality* of two quantities, or sets of quantities, is expressed by two parallel lines $=$. Thus $a+b=d$, signifies that a and b together are equal to d. So $8+4=16-4=10+2=7+2+3$.

15. When the first of the two quantities compared, is *greater* than the other, the character $>$ is placed between them. Thus $a>b$ signifies that a is greater than b.

If the first is *less* than the other, the character $<$ is used; as $a<b$; i. e. a is less than b. In both cases, the quantity towards which the character *opens*, is greater than the other.

16. A numeral figure is often prefixed to a letter. This is called a *co-efficient*. It shows how often the quantity expressed by the letter is to be taken. Thus $2b$ signifies twice b; and $9b$, 9 times b, or 9 multiplied into b.

The co-efficient may be either a whole number or a frac tion. Thus $\frac{2}{3}b$ is two thirds of b. When the co-efficient is not expressed, 1 is always to be understood. Thus a is the same as $1a$; i. e. once a.

17. The co-efficient may also be a *letter*, as well as a figure. In the quantity mb, m may be considered the co-efficient of b; because b is to be taken as many times as there are units in m. If m stands for 6, then mb is six times b. In $3abc$, 3 may be considered as the co-efficient of abc; $3a$ the co-efficient of bc; or $3ab$, the co-efficient of c.

QUEST.—How is equality represented? How inequality? What is a co-efficient? What does it show? When no co-efficient is expressed, what is understood? Is the co-efficient always a whole number? Is it always a figure?

18. A *simple* quantity is either a single letter or number, or several letters connected together *without* signs + and —. Thus a, ab, abd and $8b$, are each of them simple quantities.

19. A *compound* quantity consists of a number of simple quantities connected by the sign + or —. Thus $a+b$, $d-y$, $b-d+3h$, are each compound quantities. The members of which it is composed are called *terms*.

20. If there are *two* terms in a compound quantity, it is called a *binomial*. Thus $a+b$ and $a-b$ are binomials. The latter is also called a *residual* quantity, because it expresses the *difference* of two quantities, or the remainder, after one is taken from the other. A compound quantity consisting of *three* terms, is sometimes called a *trinomial*; one of four terms, a *quadrinomial*, &c.

21. When the several members of a compound quantity are to be subjected to the *same operation*, they must be connected by a line (——) called a *vinculum*, or by a *parenthesis* ( ). Thus $b-\overline{b+c}$, or $a-(b+c)$, shows that the *sum* of b and c is to be subtracted from a. But $a-b+c$ signifies that b only is to be subtracted from a, while c is to be added.

22. A single letter, or a number of letters, representing any quantities with their relations, is called an algebraic *expression*, or *formula*. Thus $a+b+3d$ is an algebraic expression.

23. *Multiplication* is usually denoted by two oblique lines crossing each other thus $\times$. Thus $a\times b$ is a multiplied into b: and 6×3 is 6 times 3, or 6 into 3. Sometimes a *point* is

QUEST.—What is a simple quantity? A compound? If there are two terms, what is it called? Three? Four? When several terms are subjected to the same operation, how is this shown? What is an algebraic *expression*, or *formula*? In how many ways is *multiplication* represented? First? Second? Third?

used to indicate multiplication. Thus $a \,.\, b$ is the same as $a \times b$. But the sign of multiplication is more commonly omitted, between simple quantities; and the letters are connected together in the form of a word or syllable. Thus ab is the same as $a \,.\, b$ or $a \times b$. And $bcde$ is the same as $b \times c \times d \times e$. When a compound quantity is to be multiplied, a *vinculum* or *parenthesis* is used, as in the case of subtraction. Thus the sum of a and b multiplied into the sum of c and d, is $\overline{a+b} \times \overline{c+d}$, or $(a+b) \times (c+d)$. And $(6+2) \times 5$ is 8×5, or 40. But $6+2 \times 5$ is $6+10$, or 16. When the marks of parenthesis are used, the sign of multiplication is frequently omitted. Thus $(x+y)\ (x-y)$ is $(x+y) \times (x-y)$.

24. When two or more quantities are multiplied together, each of them is called a *factor*. In the product ab, a is a factor, and so is b. In the product $x \times (a+m)$, x is one of the factors, and $a+m$ the other. Hence every *co-efficient* may be considered a factor. (Art. 17.) In the product $3y$, 3 is a factor as well as y.

25. A quantity is said to be *resolved into factors*, when any factors are taken, which, being multiplied together, will produce the given quantity. Thus $3ab$ may be resolved into the two factors $3a$ and b, because $3a \times b$ is $3ab$. And $5amn$ may be resolved into the three factors $5a$, and m, and n. And 48 may be resolved into the two factors 2×24, or 3×16, or 4×12, or 6×8; or into the three factors $2 \times 3 \times 8$, or $4 \times 6 \times 2$, &c.

26. *Division* is expressed in two ways: 1st. By a horizontal line between two dots $\div$, which shows that the quantity *preceding* it, is to be divided by that which *follows*. Thus, $a \div c$, is a divided by c.

Quest.—What is a factor? When is a quantity resolved into factors? Factors of $3ab$? $5amn$? 48? In how many ways is *division* expressed? First?

26. Division is more commonly expressed in the form of a *fraction*, putting the dividend in the place of the numerator, and the divisor in that of the denominator. Thus $\frac{a}{b}$ is a divided by b.

27. When four quantities are *proportional*, the proportion is expressed by points, in the same manner as in the Rule of Three in arithmetic. Thus $a : b :: c : d$ signifies that a has to b, the same ratio which c has to d. And $ab : cd :: a+m : b+n$, means that ab is to cd, as the sum of a and m, to the sum of b and n.

28. Algebraic quantities are said to be *alike*, when they are expressed by the same *letters*, and are of the same *power*; and *unlike*, when the letters are different, or when the same letter is raised to different powers.* Thus ab, $3ab$, $-ab$, and $-6ab$, are like quantities, because the letters are the same in each, although the signs and co-efficients are different. But $3a$, $3y$, $3bx$, are unlike quantities, because the letters are unlike, although there is no difference in the signs and co-efficients. So x, xx, and xxx, are *unlike quantities*, because they are different powers of the same quantity. (They are usually written x, x^2, and x^3.) And universally if *any quantity is repeated as a factor* a number of times in one instance, and a *different* number of times in another, the products will be *unlike quantities*; thus cc, $cccc$, and c, are unlike quantities. But if the same quantity is repeated as a factor the *same number* of times in *each instance*, the products are *like quantities*. Thus aaa, aaa, aaa, and aaa, are *like* quantities.

QUEST.—Second? The most common? How is proportion expressed? What are *like* quantities? *Unlike?* What kind of quantities are $3ab$ and $6ab$? aa and aaa? aa and aa? xx and xxx?

* For the notation of *powers* and *roots*, see sections VIII, IX.

29. One quantity is said to be a *multiple* of another, when the former *contains* the latter a certain number of times without a remainder. Thus $10a$ is a multiple of $2a$; and 24 is a multiple of 6.

30. One quantity is said to be a *measure* of another, when the former is *contained* in the latter any number of times, without a remainder. Thus $3b$ is a measure of $15b$; and 7 is a measure of 35.

31. The *value* of an expression, is the number or quantity for which the expression stands. Thus the value of $3+4$ is 7; of 3×4 is 12: of $\frac{16}{8}$ is 2.

32. *The* RECIPROCAL *of a quantity, is the quotient arising from dividing* A UNIT *by that quantity.* The reciprocal of a is $\dfrac{1}{a}$; the reciprocal of $a+b$ is $\dfrac{1}{a+b}$; the reciprocal of 4 is $\dfrac{1}{4}$.

33. What is the algebraic expression for the following statement, in which the letters a, b, c, &c., may be supposed to represent any given quantities?

Ex. 1. The product of a, b and c, divided by the difference of c and d, is equal to the sum of b and c added to 15 times h.

Ans. $\dfrac{abc}{c-d}=b+c+15h.$

2. The product of the difference of a and h into the sum of b, c and d, is equal to 37 times m, added to the quotient of b divided by the sum of h and b.

3. The sum of a and b, is to the quotient of b divided by c, as the product of a into c, to 12 times h.

4. The sum of a, b and c, divided by six times their product, is equal to four times their sum diminished by d.

5. The quotient of 6 divided by the sum of a and b, is equal to 7 times d, diminished by the quotient of b, divided by 36.

QUEST.—When is one quantity a multiple of another? When a measure? What is the value of an algebraic expression? What is the reciprocal of a quantity?

34. What will the following expressions become, when words are substituted for the signs?

6. $\frac{a+b}{h}=abc-6m+\frac{a}{a+c}$.

Ans. The sum of a and b divided by h, is equal to the product of a, b and c diminished by 6 times m, and increased by the quotient of a divided by the sum of a and c.

7. $ab+\frac{3h-c}{x+y}=d\times\overline{a+b+c}-\frac{h}{6+b}$.

8. $a+7(h+x)-\frac{c-6d}{2a+4}=(a+h)\ (b-c.)$

9. $a-b:a\ c::\frac{d-4}{m}:3\times(h+d+y)$.

10. $\frac{a-h}{2+(b-c)}+\frac{d+ab}{2m}=\frac{b\,a\times(d+h)}{a\,m}-\frac{c\,d}{h+d\,m}$.

35. At the close of an algebraic process it is often necessary to restore the numbers for which letters have been substituted at the beginning. In doing this the sign $\times$ must *not be omitted* between the numbers, as it generally is between factors expressed by letters. Thus if a stands for 3, and b for 4, the product ab is not 34, but 3×4, i. e. 12. Suppose $a=2$; $b=4$; $c=2$; $d=6$; $m=8$; and $n=10$.

Find the value of the following *algebraic expressions*.

11. $\frac{ad}{c}+a+mn=\frac{3\times6}{2}+3+8\times10=9+3+80=92$. Ans.

12. $\frac{b+mn}{cd}+\frac{bc+n}{3d}=\frac{4+80}{2\times6}+\frac{4\times2+10}{3\times6}=8$. Ans.

13. $\frac{b+ad}{c}+bcm-\frac{d\times4cn}{5a}$. 14. $bm+\frac{a\,b+3d}{d}-\frac{3bn-d\,c}{3cd}$.

15. $abm+\frac{2b}{m-b}+2n$. 16. $(a+c)\times(n-m)+\frac{m-b}{m-d}-a$.

17. $\frac{a\times(d+c)}{n-d}+abc-\frac{(c+b)\times(m-d)}{n-bc}$.

18. $\frac{ac+5m}{2n+3}+m-cb+\frac{(4d-b)\times(a-c)}{n}$.

POSITIVE AND NEGATIVE QUANTITIES.

36. A POSITIVE or AFFIRMATIVE *quantity is one which is to be added, and has the sign + prefixed to it.* (Art. 11.)

37. A NEGATIVE *quantity is one which is required to be* SUBTRACTED, *and has the sign − prefixed to it.*

When several quantities enter into a calculation, it is frequently necessary that some of them should be *added* together, while others are *subtracted.*

If, for instance, the profits of trade are the subject of calculation, and the *gain* is considered positive, the *loss* will be negative; because the latter must be *subtracted* from the former, to determine the clear profit. If the sums of a book account are brought into an algebraic process, the debt and the credit are distinguished by opposite signs.

38. The terms positive and negative, as used in the mathematics, are merely *relative.* They imply that there is, either in the nature of the quantities, or in their circumstances, or in the purposes which they are to answer in calculation, some such *opposition* as requires that one should be *subtracted* from the other. But this *opposition* is not that of *existence* and *non-existence*, nor of *one thing greater* than *nothing*, and *another less* than *nothing*. For in many cases either of the signs may be, indifferently and at pleasure, applied to the very same quantity; that is, the two characters may change places. In determining the progress of a ship, for instance, her

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QUEST.—What is a positive quantity? What sign has it? What is a negative quantity? What sign has it? In business transactions, how is the gain considered? Loss? How are the terms positive and negative used in mathematics? Imply what?

easting may be marked +, and her westing −; or the westing may be +, and the easting −. All that is necessary is, that the two signs be prefixed to the quantities, in such a manner as to show, which are to be added, and which subtracted. In different processes, they may be differently applied. On one occasion, a downward motion may be called positive, and on another occasion negative.

39. In every algebraic calculation, some one of the quantities must be fixed upon to be considered positive. All other quantities which will *increase* this, must be positive also. But those which will tend to *diminish* it, must be negative. In a mercantile concern, if the *stock* is supposed to be positive, the *profits* will be positive; for they *increase* the stock; they are to be *added* to it. But the *losses* will be negative; for they *diminish* the stock; they are to be *subtracted* from it.

40. A negative quantity is frequently *greater* than the positive one with which it is connected. But how, it may be asked, can the former be *subtracted* from the latter? The greater is certainly not *contained* in the less: how then can it be taken out of it? The answer to this is, that the greater may be supposed first to *exhaust* the less, and then to leave a remainder equal to the difference between the two. If a man has in his possession 1000 dollars, and has contracted a debt of 1500; the latter subtracted from the former, not only exhausts the whole of it, but leaves a balance of 500 against him. In common language, he is 500 dollars worse than nothing.

41. In this way, it frequently happens, in the course of an algebraic process, that a negative quantity is brought to *stand*

QUEST.—How determine which quantities are positive? Negative? Is a negative quantity ever greater than a positive, with which it is connected? How subtract the former from the latter in such a case? Give examples. Does a negative quantity ever stand alone? What denote?

alone. It has the sign of subtraction, without being connected with any other quantity, from which it is to be subtracted. This denotes that a previous subtraction has left a remainder, which is a part of the quantity subtracted. If the latitude of a ship which is 20 degrees north of the equator, is considered positive, and if she sails south 25 degrees: her motion first *diminishes* her latitude, then reduces it to *nothing*, and finally gives her 5 degrees of *south* latitude. The sign — prefixed to the 25 degrees, is retained before the 5, to show that this is what remains of the *southward* motion, after balancing the 20 degrees of north latitude.

42. A quantity is sometimes said to be *subtracted from* 0. By this is meant, that it belongs on the negative side of 0. But a quantity is said to be *added* to 0, when it belongs on the positive side. Thus, in speaking of the degrees of a thermometer, $0+6$ means 6 degrees *above* 0; and $0-6$, 6 degrees *below* 0.

AXIOMS.

43. An AXIOM is a *self-evident proposition.*

1. If the same quantity or equal quantities be *added* to equal quantities, their *sums* will be equal.

2. If the same quantity or equal quantities be *subtracted* from equal quantities, the *remainders* will be equal.

3. If equal quantities be *multiplied* into the same, or equal quantities, the *products* will be equal.

4. If equal quantities be *divided* by the same or equal quantities, the *quotients* will be equal.

5. If the same quantity be both *added to* and *subtracted from* another, the value of the latter will not be altered.

6. If a quantity be both *multiplied* and *divided* by another, the value of the former will not be altered.

QUEST.—What is meant by subtracting a quantity from 0? Added to 0? What is an axiom? Name some.

7. Quantities which are respectively equal to any other quantity, are equal to each other.

8. The whole of a quantity is greater than a part.

9. The *whole* of a quantity is equal to *all its parts.*

SECTION II.

ADDITION.

ART. 44. Ex. 1. John has x marbles and gains b marbles more. How many marbles has he in all?

In this example we wish to add x marbles to b marbles. But addition in algebra is denoted by the sign $+$. Hence $x+b$ is the answer: i. e. John has the sum of x marbles added to b marbles.

2. What is the sum of $3b$ dollars added to the sum of c dollars and f dollars?

By algebraic notation, $3b+c+f$ dollars is the answer.

45. The learner may be curious to know *how many* marbles there are in $x+b$ marbles; and *how many* dollars in $3b+c+f$ dollars? This depends upon the *number each letter stands for.* But the questions do not decide what this number is. It is not the *object*, in adding them, to ascertain the *specific* value of x and y, or of b, c, and f; but to find an algebraic expression, which will *represent* their *sum* or *amount.* This process is called *addition.* Hence

46. ADDITION *in algebra may be defined, the connecting of several quantities with their signs in one expression.*

QUEST.—How is addition denoted? Write the sum of a, b, c, and d. What is this process called? Define addition.

47. *Quantities may be added, by writing them one after another, without altering their signs.*

N. B. A *quantity* to *which* no *sign* is *prefixed* is always to be considered *positive*, i. e. the sign $+$ is understood. (Art. 12.)

What is the sum of $a+m$, and $b-8$, and $2h-3+md$?

$a+m+b-8+2h-3m+d$. Ans.

48. It is immaterial in what *order* the terms or letters are arranged. If you add 6 and 3 and 9, the amount is the same, whether you put the 6, 3, or 9 first, viz. 18. But it is frequently *more convenient* and therefore *customary* to arrange the letters alphabetically.

49. It often happens that the *expression* denoting the *sum* or *amount*, can be *simplified* by reducing several terms to one. Thus the amount $2a+7a+4a$, may be *abridged* by uniting the three terms in one. Thus $2a$ added to $7a$ is $9a$, and $4a$ added to $9a$ makes $13a$. Or $2a+7a+4a=13a$.

There are *two cases* in which *reductions* can be made.

50. Case I. When the *quantities* are *alike* and the *signs alike*, as $+4b+5b$, or $-4y-3y$, &c. *Add the co-efficients, annex the common letter or letters, and prefix the common sign.*

EXAMPLES.

1. What is the sum of $3a$, $4a$, and $6a$?

$3a+4a+6a=13a$. Ans.

2. $3xy$	3. $7b+xy$	4. $ry+3abh$	5. $cdxy+3mg$
$7xy$	$8b+3xy$	$3ry+abh$	$2cdxy+mg$
xy	$2b+2xy$	$6ry+4abh$	$5cdxy+7mg$
$2xy$	$6b+5xy$	$2ry+abh$	$7cdxy+8mg$

N. B. *The mode of proceeding is the same, when all the signs are* $-$. Thus $-3bc-bc-5bc=-9bc$.

Quest.—How add quantities? When no sign is prefixed to a quantity, what is understood? In what order are the terms or letters generally arranged? Why? Can expressions denoting the sum, ever be simplified? How? Case first?

7. $-ax$	8. $-2ab-my$	9. $-3ach-8bdy$
$-3ax$	$-ab-3my$	$-ach-bdy$
$-2ax$	$-7ab-8my$	$-5ach-7bdy$

51. CASE II. When the *quantities* are *alike*, but the *signs unlike*, as $+9b$ and $-6b$;

Take the less co-efficient from the greater; to the difference, annex the common letter or letters, and prefix the sign of the greater co-efficient.

Suppose a man's loss \$500 and his gain \$2000. The algebraic notation is $-500+2000$, i. e. \$500 is to be *subtracted* from his stock, and \$2000 *added* to it. But it will be the *same* in effect, and the expression will be *greatly abridged*, if we add the *difference* between \$500 and \$2000, viz. \$1500, to his stock.

10. What is the sum of $16ab$ and $-7ab$? Ans. $9ab$.

	11.	12.	13.	14.	15.
To	$+4b$	$5bc$	$2hm$	$-dy+6m$	$3h-dx$
Add	$-6b$	$-7bc$	$-9hm$	$4dy-m$	$5h+4dx$

53. If several positive, and several negative quantities are to be reduced to one term; first reduce those which are positive, next those which are negative, and then take the *difference* of the co-efficients of the two terms thus found.

16. Reduce $13b+6b+b-4b-5b-7b$, to one term.

$13b+6b+b=20b$; and $-4b-5b-7b=-16b$.

Then $20b-16b=4b$. Ans.

17. Add $3xy-xy+2xy-7xy+4xy-9xy+7xy-6xy$.

18. Add $3ad-6ad+ad+7ad-2ad+9ad-8ad-4ad$.

19. Add $2abm-abm+7abm-3abm+7abm$.

20. Add $axy-7axy+8axy-axy-8axy+9axy$.

QUEST.—How are like quantities added when their signs are unlike? When several positive and several negative quantities are to be reduced to one term, how proceed?

54. If *two equal* quantities have *contrary signs*, they destroy each other, and may be cancelled. Thus $+6b-6b=0$. And $(3\times 6)-18=0$, so $7bc-7bc=0$

55. *If the letters, or quantities in the several terms to be added, are* UNLIKE, *they can only be placed after each other, with their proper signs.* (Art. 47.)

21. If $4b$, and $-6y$, and $3x$, and $17h$, and $-5d$, and 6, be added; their sum will be $4b-6y+3x+17h-5d+6$.

22. Add aa, aaa, to xx, xxx and $xxxx$.

Different letters, and different powers of the same letter, can no more be united in the same term, than dollars and guineas can be added, so as to make a single sum. Six guineas and four dollars are neither ten guineas nor ten dollars.

56. From the foregoing principles we derive the following

GENERAL RULE FOR ADDITION.

Write down the quantities to be added without altering their signs, placing those that are alike under each other; and unite such terms as are similar.

23. To $3bc-6d+2b-3y$
Add $-3bc+x-3d+bg$
And $2d+y+3x+b$

These may be arranged thus:

$$\begin{array}{rrrrrrr}
3bc & -6d & +2b & -3y & & & \\
-3bc & -3d & & & +x & +bg & \\
 & 2d & & +y & +3x & & +b \\
\hline
 & -7d & +2b & -2y & +4x & +bg & +b
\end{array}$$

The sum will be $-7d+2b-2y+4x+bg+b$

EXAMPLES FOR PRACTICE.

1. Add $ab+8$, to $cd-3$, and $5ab-4m+2$.

2. Add $x+3y-dx$, to $7-x-8+hm$.

3. Add $abm-3x+bm$, to $y-x+7$, and $5x-6y+9$.

QUEST.—If two equal quantities have contrary signs, what is the effect? If the letters in the several terms are unlike, how are they added? What then is the general rule for addition?

4. Add $3am+6-7xy-8$, to $10xy-9+5am$.

5. Add $6ahy+7d-1+mxy$, to $3ahy-7d+17-mxy$.

6. Add $6ad-h+8xy-ad$, to $5ad+h-7xy$.

7. Add $3ab-2ay+x$, to $ab-ay+bx-h$.

8. Add $2by-3ax+2a$, to $3bx-by+a$.

9. Add $ax+by-xy$, to $-by+2xy+5ax$.

10. Add $41cdf-10xy-18b$, to $7xy+24b+3cdf$.

11. Add $3bz-17xy+18a$, to $4ax-5bx+63cx$.

12. Add $8ab-6bc+4cd-7xy$, to $17mn+18fg-2ax$.

13. Add $-42abc+10abd$, to $50abc+15abd+5xyz$.

14. Add $ax-y+6-df+44$, to $4df-20+3ax+75y$.

15. Add $45a-10b+4cdf$, to $82b-4cdf+10a-4b$.

16. Add $12(a+b)+3(a+b)$, to $2(a+b)-10(a+b)$.

17. Add $xy(a+b)+3xy(a+b)$, to $2xy(a+b)-4xy(a+b)$.

18. Add $ax+aa$, $x+xxx$, $4aa+2x+ax$, and $2xxx$.

19. Add $y-yy+xy$, $2xx+10yy$, to $4xy+6y-8xx$.

20. Add $aaa+4aaa$, to $10aaa-14aaa+8aaa$.

21. Add $12yyyy-10xx$, to $20xx-8yyyy+2xx+3yyyy$.

22. Add $4(x-y)-13$, to $(a+b)-16(x-y)-7(a+b)$.

23. Add $a(x+y)-6y$, to $40(a-b)+8a(x+y)-36(a-b)$.

24. Add $10axy+17bcd-axy$, to $6axy-14bcd$.

25. Add $-x+y+6x(a-b)-7x$, to $16y-15x(a-b)+25x$.

26. Add $-4(x+y)+16(x+y,)$ to $15abc-10(x+y)$.

27. Add $5abc-6xy+mn$, $a+6abc+14xy-11a+6mn$, to $15xy-17abc-15a-abc+xy-3mn+abc$.

28. Add $a(x+y)-3b(x+y)-4a(x+y)-4(x+y)-(x+y)$, to $4b(x+y)+7a(x+y)+5(x+y)+6b(x+y)$.

SECTION III.

SUBTRACTION.

ART. 57. SUBTRACTION *in algebra is finding the difference of two quantities or sets of quantities.*

1. Charles has $5a$ pears, and James has $3a$ pears. How many more has Charles than James? In this example we wish to take $3a$ pears from $5a$ pears. But subtraction in algebra is denoted by the sign $-$. Hence $5a-3a$ pears represents the answer. But $5a-3a=2a$ pears. Ans.

2. A gentleman owns a house valued at \$4500; but he is in debt \$800. How much is he worth?

\$4500−\$800=\$3700. Ans.

58. Let us now attend to the *principle* upon which this operation is performed. To illustrate this point, let us suppose that you open a book account with your neighbor. When footed up, the debtor side, which is considered positive, is \$500. The credit side, which is considered negative, is \$300. You balance the account, and find he owes you \$500 −\$300=\$200. Now if you take \$50 from the *positive* or debtor side, it will have the *same effect on the balance*, as if you add \$50 to the *negative* or credit side. On the other hand, if you take \$50 from the *negative* or credit side, it will have the same effect on the balance, as if you add \$50 to the *positive* or debtor side.

59. *Hence universally, taking away a positive quantity from an algebraic expression is the same in effect as adding an equal negative quantity; and taking away a negative quantity, is the same as adding an equal positive one.*

QUEST.—What is subtraction? On what principle is the rule founded? How illustrate this? What is the rule for subtraction?

60. Upon this principle is founded the following ind – –

GENERAL RULE FOR SUBTRACTION.

I. *Change the signs of all the quantities to be subtracted,* i. e. *of the subtrahend, or suppose them to be changed from* $+$ *to* $-$, *or from* $-$ *to* $+$.

II. *If the quantities are* ALIKE, *unite the terms as in addition.* (Arts. 50, 51.)

III. *If the quantities are* UNLIKE, *change the signs of the subtrahend, and write its terms after the minuend,* (Art. 55.) *tion.* (Art. 55.)

EXAMPLES.

1. From $6a+9b$, Take $3a+4b$. Change the signs of the subtrahend thus: $\left.\begin{array}{r} 6a+9b \\ -3a-4b \end{array}\right\}$ Ans. $3a+5b$.

2. From $16b$	3. $14da$	4. -28	5. $-16b$	6. $-14da$	
Subtr. $12b$	$6da$	-16	$-12b$	$-6da$	

7. $16b$	8. $12b$	9. $6da$	10. -16	11. $-12b$	12. $-6da$
$28b$	$16b$	$14da$	-28	$-16b$	$-14da$

14. $+16b$	15. $+14da$	16. -28	17. $-16b$	18. $-14da$
$-12b$	$-6da$	$+16$	$+12b$	$+6da$

19. From $8ab$, take $6xy$. Ans. $8ab-6xy$.

20. $6aay$	21. $16aaxx$	22. $6dd+3d-4ddd$
$17ay$	$20ax$	$10dc+2dddd+4dy$.

62. From these examples, it will be seen that the *difference* between a positive and a negative quantity, may be *greater* than either of the two quantities. In a thermometer, the difference between 28 degrees above zero, and 16 below, is 44 degrees. The difference between gaining 1000 dollars in trade, and losing 500, is equivalent to 1500 dollars.

63. PROOF.—Subtraction may be *proved*, as in arithmetic, by adding the remainder to the subtrahend. The sum ought to be equal to the minuend, upon the obvious principle, that the difference of two quantities added to one of them, is equal to the other.

23. From $2xy-1$, Sub. $-xy+7$, Rem. $3xy-8$.

Proof. $\left\{\begin{array}{ll} -xy+7 & \text{the subtrahend.} \\ 3xy-8 & \text{remainder.} \\ \hline 2xy-1 & \text{minuend.} \end{array}\right.$

24. $h+3bx$	25. $hy-ah$	26. $nd-7by$	
$3h-9bx$	$5hy-6ah$	$5nd-by$	
27.	28.	29.	30.
$3abm-\ xy$	$-17+4ax$	$ax+\ 7b$	$3ah+axy$
$-7abm+6xy$	$-20-\ ax$	$-4ax+15b$	$-7ah+axy$

65. When there are several *terms alike* in the subtrahend, they may be united and their sum be used. Thus,

31. From ab, subtract $3am+am+7am+2am+6am$.

Ans. $ab-3am-am-7am-2am-6am=ab-19am$.

32. From y, subtract $a-a-a-a$.

33. From $ax-bc+3ax+7bc$, subtract $4bc-2ax+bc+4ax$.

34. From $ad+3dc-bx$, subtract $3ad+7bx-dc+ad$.

66. The sign $-$, placed before the marks of *parenthesis* which include a number of quantities, requires, that when these marks are removed, the signs of *all* the quantities thus included should be changed.

Thus $a-(b-c+d)$ signifies that the quantities b, $-c$ and $+d$, are to be subtracted from a. Remove the ( ) and the expression will then become $a-b+c-d$.

35. $xy+d-(7ad-xy+d+hm)=-7ad+2xy-hm$.

QUEST.—How proved? When — is placed before a ( ) which includes a quantity, if the ( ) is removed, what must be done?

67. On the other hand, when a number of quantities are introduced within the marks of parenthesis, with — immediately preceding, the signs must be changed.

Thus $-m+b-dx+3h=-(m-b+dx-3h)$.

EXAMPLES FOR PRACTICE.

1. From $6ab+7xy+18dfg$; take $3xy+4ab+8dfg$.
2. From $-35ax-21ab-37m$; take $-30m-15ab-10ax$.
3. From $9ay+19bx+22bc$; take $12ay+31bc+50bx$.
4. From $8xy-10ab+6d$; take $-12ab+10d+24xy$.
5. From $7a+6x+df+xyz$; take $3x-4a-3df-17xyz$.
6. From $18bc-xy+22gh$; take $41xy-gh+bc$.
7. From $21ax+y+ac-ay$; take $4a-bc+x-yz-dc$.
8. From $21x+40xy-13a$; take $42+10ab-5bc$.
9. From $5xy$; take $2ab+30ab+ab-4ab$.
10. From $5ax+16ay$; take $4ax-ay+3ax+4ay$.
11. From $a+b$; take $-(c+d-f+g-h-xy)$.
12. From $7ab+16xy-7ad$; take $-(6ab-12xy+ad)$

13. Required to introduce the following quantities within a parenthesis with — immediately preceding, without altering their value; $-a+b-c-d+f+gh$.

14. Also, $ab-cdx+df-x-y+ghf-bc+xyz$.
15. From $4xx+6bbb$; take $3xx+4bbb$.
16. From $20yy-2y+12aaa$; take $15yy-2y-12aaa$.
17. From $-8(a+b)+10(x+y)$; take $2(a+b)-6(x+y)$.
18. From $4(a+b)-16(x-y)$; take $17(a+b)+36(x-y)$.
19. From $2a-aa+ba$; take $a-4aa-6ba$.

QUEST.—When a number of quantities are introduced within a ( ) with — before it, what must be done?

20. From $ax+3x-xxx$; take $2x+3xx+10xxx$.
21. From $18-25ab+20x+3y$; take $3x+3y-25ab+1$.
22. From $6(a-y)-17(a+y)$; take $3(a+y)-7(a-y)$.
23. From $ax-xy-my-6$; take $6ax-6xy-ay+46-7df$.
24. From $66a-4b$; take $20a-b-30a-16a-3b+5a$.

SECTION IV.

MULTIPLICATION.

Art. 68. Ex. 1. What will 4 lemons cost at x cents a piece? If 1 lemon costs x cts. 4 lemons will cost 4 times as much, i. e. $4x$ cents. Ans.

2. How much can a man earn in 5 months at a dols. per month? Reasoning as before, $a\times5=5a$ dols. Ans.

Now $4x$ is equal to $x+x+x+x$; and $5a=a+a+a+a+a$.

69. *This repeated addition of a quantity to itself is called* MULTIPLICATION.

Obs. From the definition of Multiplication, it is manifest that the *product* is a quantity of the same kind as the multiplicand.

70. *Multiplying by a whole number is taking the multiplicand as many times as there are units in the multiplier.*

Multiplying by 1, is taking the multiplicand *once*, as a.

Mult. by 2, is taking the multiplicand *twice*, as $a+a$, &c.

71. *Multiplying by a* FRACTION *is taking a certain* PORTION *of the multiplicand as many times as there are like portions of a unit in the multiplier.*

Mult. by $\frac{1}{5}$, is taking $\frac{1}{5}$ of the multiplicand *once*, as $\frac{1}{5}a$.

Mult. by $\frac{2}{5}$, is taking $\frac{1}{5}$ of the multiplicand *twice*, as $\frac{1}{5}a+\frac{1}{5}a$.

Quest.—What is multiplication? Of what denomination is the product? What is it to multiply by a whole number? By a fraction? By $\frac{2}{5}$? By $\frac{6}{8}$? By $\frac{19}{20}$?

72. *Multiplying two or more letters together, is writing them one after the other, either with, or without the sign of multiplication between them.*

Thus b multiplied into c is $b \times c$, or $b \,.\, c$, or bc. And x into y, into z, is $x \times y \times z$, or $x \,.\, y \,.\, z$, or more commonly written xyz. And am into xy is $amxy$. So abc into xyz is $abcxyz$.

73. It is immaterial as to the result *in what order* the letters are arranged. The prod. of ba is the same as ab. 3 times 5 is equal to 5 times 3. The prod. of a, b and c, is abc, or bac, or cab, or cba. It is more convenient, however, to place the letters in *alphabetical* order.

74. *When the letters have numerical* CO-EFFICIENTS, *these must be multiplied together, and prefixed to the product of the letters.*

1. Multiply $3a$ into $2b$. Ans. $6ab$. For if a into b is ab; then 3 times a into b is evidently $3ab$; and if, instead of multiplying by b, we multiply by *twice* b, the product must be *twice* as great; that is, $2 \times 3ab$, or $6ab$.

2 Mult.	$12hy$	3. $3dh$	4. $2ad$	5. $7bdh$	6. $3ay$
Into	$2rx$	my	$13hmg$	x	$8mx$

75. If either of the factors consists of figures *only*, these must be multiplied into the co-efficients and letters of the other factors.

Thus $3ab$ into 4, is $12ab$. And 36 into $2x$, is $72x$. And 24 into hy, is $24hy$.

76. If the multiplicand is a *compound* quantity, *each of its terms* must be multiplied into the multiplier. Thus $b+c+d$ into a is $ab+ac+ad$. For the whole of the multiplicand is to be taken as many times as there are units in the multiplier.

QUEST.—How are two or more letters multiplied together? In what order are they arranged? When letters have numeral co-efficients? If either factor consists of figures only? If the multiplicand is a compound quantity?

7. Mult. $d+2xy$ 8. $2h+m$ 9. $3hl+1$ 10. $2hm+3$
Into $3b$ $6dy$ my $4b$

77. N. B. The preceding instances must not be confounded with those in which several *factors* are connected by the sign $\times$, or by a point. In the latter case, the multiplier is to be written before the other factors *without being repeated.* The product of $b\times d$ into a, is $ab\times d$, and not $ab\times ad$. For $b\times d$ is bd, and this into a, is abd. (Art. 72.) The expression $b\times d$ is not to be considered like $b+d$, a *compound quantity* consisting of two terms. Different terms are always separated by $+$ or $-$. (Art. 19.) The product of $b\times h\times m\times y$ into a, is $a\times b\times h\times m\times y$, or $abhmy$. But $b+h+m+y$ into a is $ab+ah+am+ay$.

78. If *both* the factors are compound quantities, *each term in the multiplier must be multiplied into each term in the multiplicand.* Thus $(a+b)$ into $(c+d)$ is $ac+ad+bc+bd$.

For the units in the multiplier $a+b$, are equal to the units in a, *added* to the units in b. Therefore the product produced by a, must be added to the product produced by b.

$$\left.\begin{array}{l}\text{The product of } c+d \text{ into } a \text{ is } ac+ad.\\ \text{The product of } c+d \text{ into } b \text{ is } bc+bd.\end{array}\right\}\text{(Art. 76.)}$$
The product of $c+d$ into $a+b$ is therefore $ac+ad+bc+bd$.

11. Mult. $3x+d$ 12. $4ay+2b$ 13. $a+1$ 14. $2b+7$
Into $2a+hm$ $3c+rx$ $3x+4$ $6d+1$

15. Mult. $d+rx+h$ into $6m+4+7y$.

16. Mult. $7+6b+ad$ into $3r+4+2h$.

QUEST.—Does it make any difference in the result whether the quantities are connected by the sign $\times$, or $+$? If both factors are compound quantities, how proceed?

79. When several terms in the product are *alike*, it will be expedient to *set one under the other*, and then to unite them by the rules for reduction in addition. Thus,

17. Mult. $b+a$ 18. $b+c+2$ 19. $a+\ y+1$
Into $b+a$ $b+c+3$ $3b+2x+7$

$$\begin{array}{l} bb+ab \\ \quad +ab+aa \\ \hline \end{array}$$

Prod. $bb+2ab+aa$

20. Mult. $3a+d+4$ into $2a+3d+1$.
21. Mult. $b+cd+2$ into $3b+4cd+7$.
22. Mult. $3b+2x+h$ into $a\times d\times 2x$.

80. It will be easy to see that when the multiplier and multiplicand consist of any quantity, *repeated as a factor*, this factor will be repeated in the product as many times as in the multiplier and multiplicand together. (Art. 163, 3.)

23. Mult. $a\times a\times a$. Here a is repeated *three times* as a factor.
Into $a\times a$ Here it is repeated *twice*.

Prod. $a\times a\times a\times a\times a$. Here it is repeated *five times*.

34. What is the product of $bbbb$ into bbb?
25. What is the product of $2x\times 3x\times 4x$ into $5x\times 6x$?

81. But the *numeral co-efficients* of several fellow-factors should be brought together by multiplication. Thus

26. $2a\times 3b$ into $4a\times 5b$ is $2a\times 3b\times 4a\times 5b$, or $120aabb$.

For the co-efficients are *factors*, (Art. 24,) and it is immaterial in what *order* these are arranged. (Art. 73.) So that

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QUEST.—When several terms in the product are alike, how proceed? When the multiplier and multiplicand consist of the same factor repeated, how many times will it be repeated in the product? What should be done with numeral co-efficients?

$2a\times3b\times4a\times5b=2\times3\times4\times5\times a\times a\times b\times b=120aabb$.

27. The product of $3a\times4bh$ into $5m\times6y$.

28. The product of $4b\times6d$ into $2x+1$.

RULE FOR SIGNS IN THE PRODUCT.

82. $+$ into $+$ produces $+$; $-$ into $+$ gives $-$; $+$ into $-$ gives $-$; and $-$ into $-$ gives $+$. That is, *if the signs of the factors are* ALIKE, *the sign of the product will be affirmative; but if the signs of the factors are* UNLIKE, *the sign of the pro uct will be negative.*

83. The first case, that of $+$ into $+$, needs no farther illustration. The second is $-$ into $+$, that is, the multiplicand is negative, and the multiplier positive. Thus, $-a$ into $+4$ is $-4a$. For the repetitions of the multiplicand are, $-a-a-a-a=-4a$.

30. Mult.	$2a-m$	31. $h-3d+4$	32. $a-2-7d-x$
Into	$3h+x$	$2y$	$3b+h$

84. In the two preceding cases, the positive sign prefixed to the multiplier shows, that the repetitions of the multiplicand are to be *added* to the other quantities with which the multiplier is connected. But in the two remaining cases, the negative sign prefixed to the multiplier, indicates that the sum of the repetitions of the multiplicand are to be *subtracted* from the other quantities. (Art. 70, 71.)

Obser. This subtraction is performed, at the time of multiplying, by making the sign of the product opposite to that of the multiplicand. Thus $+a$ into -4 is $-4a$. For the repetitions of the multiplicand are, $+a+a+a+a=+4a$.

QUEST.—Rule for the signs? When the multiplicand is $+$, what does it show? When $-$, what? When and how is the subtraction performed?

But this sum is to be *subtracted* from the other quantities with which the multiplier is connected. It will then become $-4a$. (Art. 59.)

Thus in the expression $b-(4\times a)$ it is manifest that $4\times a$ is to be subtracted from b. Now $4\times a$ is $4a$, that is $+4a$. But to subtract this from b, the sign $+$ must be changed into $-$. So that $b-(4\times a)$ is $b-4a$. And $a\times-4$ is therefore $-4a$.

Again, suppose the multiplicand is a, and the multiplier $(6-4)$. As $(6-4)$ is equal to 2, the product will be equal to $2a$. This is *less* than the product of 6 into a. To obtain then the product of the compound multiplier $(6-4)$ into a, we must *subtract* the product of the negative part from that of the positive part.

33. $$\left.\begin{array}{lr}\text{Multiplying} & a\\ \text{Into} & 6-4\\ \hline\end{array}\right\}\text{ is the same as }\left\{\begin{array}{lr}\text{Multiplying} & a\\ \text{Into} & 2\\ \hline\end{array}\right.$$

And the product $6a-4a$, is the same as the product, $2a$.

But if the multiplier had been $(6+4)$, the two products must have been *added*.

34. $$\left.\begin{array}{lr}\text{Multiplying} & a\\ \text{Into} & 6+4\\ \hline\end{array}\right\}\text{ is the same as }\left\{\begin{array}{lr}\text{Multiplying} & a\\ \text{Into} & 10\\ \hline\end{array}\right.$$

And the product $6a+4a$, is the same as the product $10a$.

N. B. This shows at once the difference between multiplying by a *positive* factor, and multiplying by a *negative* one. In the former case, the sum of the repetitions of the multiplicand is to be *added to*, in the latter *subtracted from*, the other quantities, with which the multiplier is connected. (Art. 41.)

QUEST.—What is the difference between multiplying by a positive factor and a negative one?

36. Mult. $a+x$ 37. $3dy+hx+2$ 38. $3h+3$
Into $b-x$ $mr-ab$ $ad-6$

85. If *two negatives* be multiplied together, the product will be affirmative: $-4\times-a=+4a$. In this case, as in the preceding, the repetitions of the multiplicand are to be *subtracted*, because the multiplier has the negative sign. These repetitions, if the multiplicand is $-a$, and the multiplier -4, are $-a-a-a-a=-4a$. But this is to be subtracted by changing the sign. It then becomes $+4a$.

Suppose $-a$ is multiplied into $(6-4)$. As $6-4=2$, the product is, evidently, *twice* the multiplicand, that is, $-2a$. But if we multiply $-a$ into 6 and 4 separately; $-a$ into 6 is $-6a$, and $-a$ into 4 is $-4a$. (Art. 83.) As in the multiplier, 4 is to be subtracted from 6; so, in the product, $-4a$ must be subtracted from $-6a$. Now $-4a$ becomes by subtraction $+4a$. The whole product then is $-6a+4a$, which is equal to $-2a$. Or thus,

39. Multiplying $-a$ Into $6-4$ } is the same as { Multiplying $-a$ Into 2

And the prod. $-6a+4a$, is equal to the product $-2a$.

It is often considered a great mystery, that the product of two negatives should be affirmative. But it amounts to nothing more than this, that the subtraction of a negative quantity is equivalent to the addition of an affirmative one, (Art. 58, 59;) and, therefore, that the *repeated* subtraction of a negative quantity, is equivalent to a *repeated* addition of an affirmative one. Taking off from a man's hands a debt of ten dollars every month, is adding ten dollars a month to the value of his property.

QUEST.—Explain how — into — gives +.

40. Multiply $a-4$ into $3b-6$.

41. Mult. $3ad-ah-7$ into $4-dy-hr$.

42. Mult. $2hy+3m-1$ into $4d-2x+3$.

86. Positive and negative terms may frequently *balance* each other, so as to disappear in the product. (Art. 54.)

43. Mult. $a-b$ | 44. $mm-yy$ | 45. $aa+ab+bb$
Into $a+b$ | $mm+yy$ | $a-b$

$$\begin{array}{rl} & aa-ab \\ & \quad +ab-bb \\ \hline \text{Prod.} & aa \quad * \quad -bb \end{array}$$

87. For many purposes, it is sufficient merely to *indicate* the multiplication of compound quantities, without actually multiplying the several terms. Thus (Art. 23,) the product of

$a+b+c$ into $h+m+y$, is $(a+b+c)\times(h+m+y)$.

47. What is the product of $a+m$ into $h+x$ and $d+y$?

By this method of representing multiplication, an important advantage is often gained, in preserving the factors distinct from each other.

When the several terms are multiplied in form, the expression is said to be *expanded*.

48. What does $(a+b)\times(c+d)$ become when expanded?

89. With a given multiplicand, the less the multiplier, the less will be the product. If then the multiplier be reduced to *nothing*, the *product* will be *nothing*. Thus $a\times0=0$. And if 0 be one of *any number* of fellow-factors, the product of the whole will be nothing.

QUEST.—Is it always necessary actually to perform the multiplication? What advantage is gained by representing it? When is an expression said to be *expanded*? When you multiply a quantity by 0, what is the product?

49. What is the product of $ab \times c \times 3d \times 0$?

50. And $(a+b)\times(c+d)\times(h-m)\times 0$?

From the preceding principles we derive the following

GENERAL RULE FOR MULTIPLICATION.

90. *Multiply the letters and coëfficients of each term in the multiplicand, into the letters and co-efficients of each term in the multiplier; and prefix to each term of the product, the sign required by the principle, that like signs produce +, and unlike signs —.*

EXAMPLES FOR PRACTICE.

1. Mult. $a+3b-2$ into $4a-6b-4$.
2. Mult. $4ab\times x\times 2$ into $3my-1+h$.
3. Mult. $(7ah-y)\times 4$ into $4x\times 3\times 5\times d$.
4. Mult. $(6ab-hd+1)\times 2$ into $(8+4x-1)\times d$.
5. Mult. $3ay+y-4+h$ into $(d+x)\times(h+y)$.
6. Mult. $6ax-(4h-d)$ into $(b+1)\times(h+1)$.
7. Mult. $7ay-1+h\times(d-x)$ into $-(r+3-4m)$.
8. Mult. $a+b$ into $a+b$ into $a+b$.
9. Mult. $x+y$ into $x-y$ into $x+y$.
10. Mult. $aa+bb$ into $cc+dd$ into $xx+yy$.
11. Mult. $abc-def+x-7+y$ into $a+b$.
12. Mult. $xy-yy+10$ into $aa-12$.
13. Mult. $4(x+y)$ into $3a$ into $6b$ into 3.
14. Mult. $3(a+b+c+d)$ into xyz, into m.
15. Mult. $a-b-c+d$ into $6\times(c+d)$.
16. Mult. $xx+xy+yy$ into $x-y$.

QUEST.—What is the general rule for multiplication?

17. Mult. $aaa-bbb$ into $aaa+bbb$.

18. Mult. $aa-ax+xx$ into $a+x$.

19. Mult. $yyy-ayy+aay-aaa$ into $y+a$.

20. Mult. $15a+20bb$ into $3a-4bb$.

21. Mult. $3a(x+y)\times 4$ into $a+b$.

22. Mult. $21xy-18a+2-7c$ into $1-x$.

23. Mult. $2ax-y$ into $-(b+2)$ into xyz.

24. Mult. $25+6ab$ into $-(x-y)$ into $-2+m$.

25. Mult. $aa+2ab+bb$ into $a+b$ into $a+b$.

SECTION V.

DIVISION.

ART. 91. PROB. 1. A man divided $48x$ peaches among 6 boys. How many did each receive?

If 6 boys receive $48x$ peaches, it is manifest 1 boy will receive $\frac{1}{6}$ of $48x$ peaches; but $\frac{1}{6}$ of $48x=48x\div 6=8x$ peaches. Ans.

2. If 8 hats cost $24a$ dollars, what will 1 hat cost?

Reasoning as before, 1 hat will cost $\frac{1}{8}$ of $24a$ dollars, viz. $2a$ dollars. Ans.

This process is called DIVISION. *It consists in finding how many times one quantity contains another;* and is the *reverse* of multiplication. The quantity to be divided is called the *dividend;* the *given* factor, the *divisor;* and that which is *required*, the *quotient.* Hence,

---

QUEST.—What is division? Of what, the reverse? The quantity to be divided, called? To divide by? The quantity sought?

DIVISION *is finding a quotient, which multiplied into the divisor will produce the dividend.*

92. As the product of the divisor and quotient is equal to the dividend, the quotient may be found, by resolving the dividend into two such factors, that one of them shall be the divisor. The other will, of course, be the quotient.

Suppose abd is to be divided by a. The factors a and bd will produce the dividend. The first of these, being a divisor, may be set aside. The other is the quotient. Hence,

When the divisor is found as a factor in the dividend, the division is performed by cancelling this factor.

1. Divide cx by c. Ans. x. 2. Divide dh by d.

3. Divide	drx	4. hmy	5. $dhxy$	6. $abcd$	7. $abxy$
By	dr	hm	dy	b	ax

93. PROOF.—*Multiply the divisor and quotient together, and the product will be equal to the dividend, if the work is right.*

Thus $ax \div a$ the quotient is x. Proof. $x \times a = ax$, dividend.

94. If a letter is *repeated* in the dividend, care must be taken that the factor rejected be only equal to the divisor.

8. Divide aab by a. Ans. ab. 9. Divide bbx by b.

10. Div.	$aadddx$	11. $aammyy$	12. $aaaxxxh$	13. yyy
By	ad	amy	$aaxx$	yy

In such instances, it is obvious that we are not to reject *every* letter in the dividend which is the same with one in the divisor.

95. If the dividend consists of *any factors whatever*, expunging one of them is dividing by it.

QUEST.—When the divisor is a factor of the dividend, how proceed? Proof? If the letter is repeated in the dividend, what is necessary? If the dividend consist of any factors, what effect has expunging one of them?

14. Divide $a(b+d)$ by a. Ans. $b+d$.

15. Div. $a(b+d)$	16. $(b+x)(c+d)$	17. $(b+y)\times(d-h)x$
By $b+d$	$b+x$	$d-h$

96. If there are *numeral co-efficients prefixed to the letters, the co-efficients of the dividend must be divided by the co-efficients of the divisor.*

18. Divide $6ab$ by $2b$. Ans. $3a$. 19. Div. $16dxy$ by $4dx$.

20. Div. $25dhr$	21. $12xy$	22. $34drx$	23. $20hm$
By dh	6	34	m

97. When a simple factor is multiplied into a *compound* one, the former enters into *every* term of the latter. (Art. 76.) Thus a into $b+d$, is $ab+ad$. Such a product is easily resolved again into its original factors. Thus $ab+ad= a\times(b+d)$.

25. $ab+ac+ah=a\times(b+c+h)$.

26. What are the factors of $amh+amx+amy$?

27. What are the factors of $4ad+8ah+12am+4ay$?

Now if the whole quantity be divided by one of these factors, according to Art. 95, the quotient will be the other factor.

Thus, $(ab+ad)\div a=b+d$.

29. $(ab+ad)\div(b+d)=a$. Hence,

If the divisor is contained in *every* term of a *compound* dividend, *it must be cancelled in each.*

	30.	31.	32.	33.
Div.	$ab+ac$	$bdh+bdy$	$aah+ay$	$drx+dhx+dxy$
By	a	b	a	dx

QUEST.—If there are numeral co-efficients, how proceed? When the divisor is contained in every term of a compound dividend, how proceed?

	34.	35.	36.	37.
Div.	$6ab+12ac$	$10dry+16d$	$12hx+8$	$35dm+14dx$
By	$3a$	$2d$	4	$7d$

98. On the other hand, *if a compound expression containing any factor in every term, be divided by the other quantities connected by their signs, the quotient will be that factor.* See the first part of the preceding article.

	38.	39.	40.	41.
Div.	$ab+ac+ah$	$amh+amx+amy$	$4ab+8ay$	$ahm+ahy$
By	$b+c+h$	$h+x+y$	$b+2y$	$m+y$

99. In division, as well as in multiplication, the caution must be observed, not to confound *terms* with *factors.* (Art. 77.)

42. Thus $(ab+ac)\div a=b+c$. (Art. 97.)
43. But $(ab\times ac)\div a=aabc\div a=abc$.
44. Quot. of $(ab+ac)\div(b+c)$. 45. And $ab\times ac\div(b\times c)$.

100. Signs.—*In division, the same rule is to be observed respecting the signs, as in multiplication; that is, if the divisor and dividend are both positive, or both negative, the quotient must be positive: if one is positive and the other negative, the quotient must be negative.* (Art. 82.)

This is manifest from the consideration that the product of the divisor and quotient must be the same as the dividend.

$$\left.\begin{array}{ll} \text{46. If} & +a\times+b=+ab \\ \text{47. “} & -a\times+b=-ab \\ \text{48. “} & +a\times-b=-ab \\ \text{49. “} & -a\times-b=+ab \end{array}\right\}\text{ then }\left\{\begin{array}{l} +ab\div+b=+a \\ -ab\div+b=-a \\ -ab\div-b=+a \\ +ab\div-b=-a \end{array}\right.$$

Quest.—What caution as to *terms* and *factors?* The rule for the signs?

	50.	51.	52.	53.
Div.	abx	$8a-10ay$	$3ax-6ay$	$6am\times dh$
By	$-a$	$-2a$	$3a$	$-2a$

101. *If the letters of the divisor are not to be found in the dividend, the division is expressed by writing the divisor under the dividend in the form of a vulgar fraction.*

54. Thus $xy\div a=\frac{xy}{a}$. 55. $(d-x)\div -h=\frac{d-x}{-h}$.

This is a method of *denoting* division, rather than an actual performing of the operation. But the purposes of division may frequently be answered by these fractional expressions. As they are of the same nature with other vulgar fractions, they may be added, subtracted, multiplied, &c.

102. If *some* of the letters in the divisor are in each term of the dividend, the fractional expression may be rendered more simple, by rejecting equal factors from the numerator and denominator.

	56.	57.	58.	59.	60.
Div.	ab	dhx	$ahm-3ay$	$ab+bx$	$2am$
By	ac	dy	ab	by	$2xy$

N. B. These reductions are made upon the principle, that a given divisor is contained in a given dividend, just as many times as *double* the divisor in *double* the dividend; *triple* the divisor in *triple* the dividend, &c.

103. If the divisor is in some of the terms of the dividend, but not in all, those which contain the divisor may be divided as in Art. 92, and the others set down in the form of a fraction.

QUEST.—If the letters of the divisor are not found in the dividend, how proceed? If some of the letters in the divisor are found in each term of the dividend? If the divisor is in some of the terms of the dividend, but not in all?

61. Thus $(ab+d)\div a$ is either $\frac{ab+d}{a}$ or $\frac{ab}{a}+\frac{d}{a}$ or $b+\frac{d}{a}$.

	62.	63.	64.	65.
Div.	$dxy+rx-hd$	$2ah+ad+x$	$bm+3y$	$2my+dh$
By	x	a	$-b$	$2m$

104. The quotient of any quantity divided by *itself* or *its equal*, is obviously *a unit*. Thus $\frac{a}{a}=1$.

67. Div. $\frac{3ax}{3ax}$. 68. $\frac{6}{4+2}$ 69. $\frac{a+b-3h}{a+b-3h}$.

	70.	71.	72.	73.
Div.	$ax+x$	$3bd-3d$	$4axy-4a+8ad$	$3ab+3-6m$
By	x	$3d$	$4a$	3

Cor. If the dividend is *greater* than the divisor, the quotient must be *greater than a unit:* but if the dividend is *less* than the divisor, the quotient must be *less than a unit.*

74. Divide 25 by 5. Ans. 5. 75. $\frac{4}{20}=\frac{1}{5}$ Ans.

DIVISION BY COMPOUND DIVISORS.*

105. Ex. 1. Divide $ac+bc+ad+bd$, by $a+b$.

$$
\begin{array}{l}
a+b)ac+bc+ad+bd(c+d \\
\quad\quad ac+bc\text{, the first subtrahend.} \\
\quad\quad \overline{\ *\quad *\ } \\
\quad\quad\quad\quad ad+bd \\
\quad\quad\quad\quad ad+bd\text{, the second subtrahend.} \\
\quad\quad\quad\quad \overline{\ *\quad *\ }
\end{array}
$$

QUEST.—To what is the quotient of any quantity divided by itself, equal? Corollary?

* The reason for inserting this article in the present place, may be learned from the preface.

Here ac, the first term of the dividend, is divided by a, the first term of the divisor, (Art. 92,) which gives c for the first term of the quotient. Multiplying the whole divisor by this, we have $ac+bc$ to be subtracted from the two first terms of the dividend. The two remaining terms are then brought down, and the first of them is divided by the first term of the divisor as before. This gives d for the second term of the quotient. Then multiplying the divisor by d, we have $ad+bd$ to be subtracted, which exhausts the whole dividend without leaving any remainder. (Art. 98.)

The rule is founded on this principle, that the product of the divisor into the several parts of the quotient, is equal to the dividend. (Art. 91.)

106. *Before beginning to divide, the terms should be so arranged that the letter, which is in the first term of the divisor, shall also be in the first term of the dividend. If this letter is repeated as a factor, either in the divisor, or dividend, or in both, the terms should be arranged in the following order; put that term first, which contains this letter the greatest number of times; the term containing it the next greatest number of times, next, and so on.*

2. Divide $2aab+bbb+3abb+aaa$ by $aa+bb+ab$.

If we take aa for the *first* term of the *divisor*, the other terms must be arranged according to the number of times a is repeated as a factor in each. Thus,

$$aa+ab+bb)aaa+2aab+2abb+bbb(a+b$$

$$\begin{array}{l} aaa+aab+abb \\ \hline \qquad aab+abb+bbb \\ \qquad aab+abb+bbb \\ \hline \qquad * \quad * \quad * \end{array}$$

QUEST.—When the divisor and dividend are both compound quantities, how arrange the terms?

N. B. The *strictest attention* must be paid to the *rules* for the *signs* in subtraction, multiplication, and division. (Arts. 60, 82, 100.)

3. Divide $xx-2xy+yy$, by $x-y$.
4. Divide $aa-bb$, by $a+b$.
5. Divide $bb+2bc+cc$, by $b+c$.
6. Divide $aaa+xxx$, by $a+x$.
7. Divide $2ax-2aax-3aaxy+6aaax+axy-xy$, by $2a-y$.
8. Divide $a+b-c-ax-bx+cx$, by $a+b-c$.
9. Divide $ac+bc+ad+bd+x$, by $a+b$.
10. Divide $ad-ah+bd-bh+y$, by $d-h$.

107. From the preceding principles we derive the following

GENERAL RULE FOR DIVISION.

I. Division, *in all cases, may be expressed by writing the divisor under the dividend in the form of a fraction.*

II. *When the divisor and dividend are both simple quantities, and have letters or factors common to each; divide the co-efficient of the divisor by that of the dividend, and cancel the factors in the dividend which are equal to those in the divisor.*

III. *When the divisor is a simple, and the dividend a compound quantity; divide each term of the dividend by the divisor as before; setting down those terms which cannot be divided in the form of a fraction.*

IV. *If the divisor and dividend are both compound quantities, arrange the terms according to Art.* 106.

To obtain the first term in the quotient, divide the first term of the dividend by the first term of the divisor. Multiply the whole divisor by the term placed in the quotient; subtract the

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Quest.—What is the general rule for division?

product from the dividend; and to the remainder, bring down as many of the following terms as shall be necessary to continue the operation. Divide again by the first term of the divisor, and proceed as before, till all the terms of the dividend are brought down.

V. SIGNS.—*If the signs in the divisor and dividend are* ALIKE, *the quotient will be* $+$; *if* UNLIKE, *the quotient will be* $-$.

EXAMPLES FOR PRACTICE.

1. Divide $12aby+6abx-18bbm+24b$, by $6b$.
2. Divide $16a-12+8y+4-20adx+m$, by 4.
3. Divide $(a-2h)\times(3m+y)\times x$, by $(a-2h)\times(3m+y.)$
4. Divide $ahd+4ad+3ay-a$, by $hd-4d+3y-1$.
5. Divide $ax-ry+ad-4my-6+a$, by $-a$.
6. Divide $amy+3my-mxy+am-d$, by $-dmy$.
7. Divide $ard-6a+2r-hd+6$, by $2ard$.
8. Divide $6ax-8+2xy+4-6hy$, by $4axy$.
9. Divide $16abcx-12xyab+24abxd-36ahgb$, by $4ab$.
10. Divide $21aaby+42cdxaa+14aaa-35aaaab$, by $7aa$.
11. Divide $12abxyz-6hdabxy+24xyabm$, by $3abxy$.
12. Divide $3ax-36bx+42-72cx+30ax$, by $3x$.
13. Divide $40ab-4(x+y)+72+12(a+b)+48c$, by -4.
14. Divide $abx-cdx+8gx+x$, by $ab-cd+8g+1$.
15. Divide $24xyz-36cd-48abcd$, by $12xyz-18cd-24abcd$.
16. Divide $-ab-ad+ax(a+b)-42axy+ab$, by $-a$.
17. Divide $6am-10ah+20-12cd+17a$, by $-2am$.
18. Divide $xyz+6x+2z-1+2xyz(a+b)$, by $6xyz$.
19. Divide $-6ac-12bc-6ab-10-2aabbcc$, by $-6abc$.
20. Divide $18abyx+16abx-20bbcm+24ab$, by $2b$.

21. Divide $16x-24+8a+84-20ax-a$, by -4.
22. Divide $(x-y)\times(3a+x)\times b$, by $(x-y)\times(3a+x)$.
23. Divide $41d\times(4-a)\times(x+y)$, by $(4-a)\times 41d$.
24. Divide $-40xy+7abx-3ahmx$, by $-40y+7ab-3ahm$.
25. Divide $20(ab+1)-60(ab+1)+50(ab+1)$, by $5a$.

Examples of Compound Quantities.

26. Divide $6ax+2xy-3ab-by+3ac+cy+h$, by $3a+y$.
27. Divide $aab-3aa+2ab-6a-4b+22$, by $b-3$.
28. Divide $bb+3bc+2cc$, by $b+c$.
29. Divide $8aaa-bbb$, by $2a-b$.
30. Divide $xxx-3axx+3aax-aaa$, by $x-a$.
31. Divide $2yyy-19yy+26y-16$, by $y-8$.
32. Divide $xxxxxx-1$, by $x-1$.
33. Divide $4xxxx-9xx+6x-3$, by $2xx+3x-1$.

NOTE.—For examples in dividing compound quantities in which the indices are used, see Art. 194, Exs. 23–40, and Art. 196.

SECTION VI.

FRACTIONS.

ART. 108. FRACTIONS *in algebra*, as well as in arithmetic, have reference to *parts* of numbers or quantities. The term is derived from a Latin word, which signifies *broken*. Thus $\frac{a}{2}$ is $\frac{1}{2}a$; $\frac{b}{4}$ is $\frac{1}{4}b$; $\frac{2a}{3}$ is $\frac{2}{3}a$; and $\frac{4x}{7}$ is $\frac{4}{7}x$.

QUEST.—What are fractions? From what is the term derived? The meaning of it?

109. Expressions in the form of fractions occur more frequently in algebra than in arithmetic. Most instances in division belong to this class. Indeed the numerator of *every* fraction may be considered as a *dividend*, of which the denominator is a *divisor*.

110. The *value* of a fraction, is the *quotient* of the numerator divided by the denominator.

Thus the value of $\frac{6}{2}$ is 3. The value of $\frac{ab}{b}$ is a.

111. From this it is evident, that whatever changes are made in the *terms* of a fraction, if the *quotient* is not altered, the value remains the same. For any fraction, therefore, we may substitute any *other* fraction which will give the same quotient.

Thus $\frac{4}{2}=\frac{10}{5}=\frac{4ba}{2ba}=\frac{8drx}{4drx}=\frac{6+2}{3+1}$, &c. For the quotient in each of these instances is 2.

112. It is also evident from the preceding articles, that *if the numerator and denominator are both multiplied, or both divided, by the same quantity, the value of the fraction will not be altered.*

Thus $\frac{3}{4}=\frac{27}{36}$, each term being multiplied by 9; and $\frac{27}{36}=\frac{9}{12}=\frac{3}{4}$, each term being divided by 3, and that quotient by 3 again. So $\frac{bx}{b}=\frac{abx}{ab}=\frac{3bx}{3b}=\frac{\frac{1}{2}bx}{\frac{1}{2}b}=\frac{\frac{1}{3}abx}{\frac{1}{3}ab}$; for the quotient in each case is x.

113. Any integral quantity may, without altering its value, be thrown into the form of a fraction, *by making* 1 *the denominator; or by multiplying the quantity into any proposed*

QUEST.—Are fractions in arithmetic or algebra the most frequent? When division is expressed in the form of a fraction, where do you place the divisor? What is the value of a fraction? If the numerator and denominator are both multiplied, or both divided by the same quantity, how is the value affected? How put an integer into the form of a fraction?

denominator, and the product will be the numerator of the fraction required.

Thus $a=\frac{a}{1}=\frac{ab}{b}=\frac{ad+ah}{d+h}=\frac{6adh}{6dh}$. The quot. of each is a.

So $d+h=\frac{dx+hx}{x}$. And $r+1=\frac{2drr+2dr}{2dr}$

114. Signs.—(1.) *Each sign* in the numerator and denominator of a fraction, affects only the *single* term to which it is prefixed.

(2.) The *dividing line* answers the purpose of a *vinculum*, i. e. it connects the several terms of which the numerator and denominator may each be composed.

The *sign prefixed* to it, therefore, affects the *whole* fraction collectively. It shows that the *value* of the whole fraction is to be subjected to the operation denoted by this sign.

(3.) Hence, *if the sign before the dividing line is changed from + to —, or from — to +, the value of the whole fraction is also changed.*

The value of $\frac{ab}{b}$ is a. (Art 110.) But this will become *negative* if the sign — is prefixed to the fraction. Thus, $y+\frac{ab}{b}=y+a$. But $y-\frac{ab}{b}=y-a$.

Note.—There is frequent occasion to remove the denominator; also to incorporate a fraction with an integer, or with another fraction. In *each* of these cases, if the sign — is prefixed to the dividing line, *all the signs of the numerator*

Quest.—How far does the effect of each sign in the numerator and denominator extend? How far, the sign prefixed to the dividing line? What does it show? When this sign is changed, what is the effect? If the sign — is prefixed to the fraction, and you wish to remove the denominator, or to incorporate the fraction with an integer, or with another, what must you do?

must be changed, as in Art. 66, where a parenthesis, having the sign $-$ before it, is removed.

Thus $b-\frac{ad+ah}{a}$ is $b-d-h$; and $b-\frac{ad-ah}{a}=b-d+h$.

(4.) *If all the signs of the numerator are changed, the value of the fraction is changed in the same manner.*

Thus $\frac{ab}{b}=+a$, (Art. 100;) but $\frac{-ab}{b}=-a$. And $\frac{ab-bc}{b}=$ $a-c$; but $\frac{-ab+bc}{b}=-a+c$.

(5.) *Again, if all the signs of the denominator are changed, the value is also changed.*

Thus $\frac{ab}{b}=+a$; but $\frac{ab}{-b}=-a$.

115. *If then the sign prefixed to a fraction, or all the signs of the numerator, or all the signs of the denominator, be changed; the value of the fraction will be changed, from positive to negative, or from negative to positive.*

116. *If any two of these changes are made at the same time, they will balance each other, and the value of the fraction will not be altered.*

Thus by changing the sign of the numerator,

$$\frac{ab}{b}=+a \text{ becomes } \frac{-ab}{b}=-a.$$

But by changing both the numerator and denominator, it becomes $\frac{-ab}{-b}=+a$, where the positive value is restored.

By changing the sign before the fraction,

$$y+\frac{ab}{b}=y+a \text{ becomes } y-\frac{ab}{b}=y-a.$$

QUEST.—If all the signs of the numerator, or of the denominator, or the sign before the fraction are changed, what is the effect? What is the effect when any two of these changes are made at the same time?

But by changing the sign of the numerator also, it becomes $y - \frac{-ab}{b}$, where the quotient $-a$ is to be subtracted from y, or which is the same thing, (Art. 59,) $+a$ is to be *added*, making $y+a$ as at first.

So $\frac{6}{2} = \frac{-6}{-2} = -\frac{-6}{2} = -\frac{6}{-2} = +3$.

And $\frac{6}{-2} = \frac{-6}{2} = -\frac{6}{2} = -\frac{-6}{-2} = -3$.

Hence the quotient in division may be set down in different ways. Thus $(a-c) \div b$, is either $\frac{a}{b} + \frac{-c}{b}$, or $\frac{a}{b} - \frac{c}{b}$.

The latter method is the most common. See the examples in Art. 103.

REDUCTION OF FRACTIONS.

117. A FRACTION *may be reduced to lower terms, by dividing both the numerator and denominator, by any quantity which will divide them without a remainder.*

According to Art. 112 this will not alter the *value* of the fraction.

1. Reduce $\frac{ab}{cb}$ to lower terms. Ans. $\frac{a}{c}$.

2. Reduce $\frac{6dm}{8dy}$. 3. Reduce $\frac{7m}{7mr}$.

4. Reduce $\frac{a+bc}{(a+bc) \times m}$. 5. Reduce $\frac{am+ay}{bm+by}$.

N. B. If a letter is in *every* term, both of the numerator and denominator, it may be *cancelled*, for this is *dividing* by that letter. (Art. 97.) Thus,

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QUEST.—How reduce a fraction to lower terms?

6. Reduce $\frac{3am+ay}{ad+ah}$. Ans. $\frac{3m+y}{d+h}$. 7. Reduce $\frac{dry+dy}{dhy-dy}$.

If the numerator and denominator be divided by the *greatest common measure*, it is evident that the fraction will be reduced to the *lowest* terms. For the method of finding the greatest common measure, see Art. 195, a.

118. To reduce fractions of different denominators to a common denominator.

Multiply each numerator into all the denominators except its own for a new numerator; and all the denominators together, for a common denominator.

8. Reduce $\frac{a}{b}$, and $\frac{c}{d}$, and $\frac{m}{y}$ to a common denominator.

$$\left.\begin{array}{l} a\times d\times y=ady \\ c\times b\times y=cby \\ m\times b\times d=mbd \end{array}\right\}\text{the three numerators.}$$

$$b\times d\times y=bdy \quad \text{the common denominator.}$$

The fractions reduced are $\frac{ady}{bdy}$, $\frac{bcy}{bdy}$ and $\frac{bdm}{bdy}$.

N. B. It will be seen that the reduction consists in multiplying the numerator and denominator of each fraction, into all the other denominators. This does not alter the value. (Art. 112.)

9. Reduce $\frac{dr}{3m}$, and $\frac{2h}{g}$, and $\frac{6c}{y}$ to a common denom.

10. Reduce $\frac{2}{3}$, and $\frac{a}{x}$, and $\frac{r+1}{d+h}$ to a common denom.

11. Reduce $\frac{1}{a+b}$, and $\frac{1}{a-b}$ to a common denom.

QUEST.—How to a common denominator? Does this alter the value of each fraction? Why not?

An *integer* and a fraction are easily reduced to a common denominator. (Art. 113.)

12. Thus a and $\frac{b}{c}$ are equal to $\frac{a}{1}$ and $\frac{b}{c}$, or $\frac{ac}{c}$ and $\frac{b}{c}$.

13. Reduce $a, b, \frac{h}{m}, \frac{d}{y}$.

14. Reduce $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$.

15. Reduce $\frac{3x}{a}, \frac{y}{5b}, \frac{1}{2}$.

16. Reduce $b, \frac{x}{y}, \frac{c}{2}$.

17. Reduce $\frac{x}{a}, \frac{b}{x}, \frac{3a}{y}, \frac{1}{3}$.

18. Reduce $\frac{3x}{a}, \frac{b}{4c}, \frac{x}{5}$.

19. Reduce $\frac{a}{b}, \frac{5}{7}, \frac{8a}{y}, \frac{1}{2}$.

20. Reduce $\frac{4a}{x}, 17, \frac{y}{c}, x, \frac{c}{4a}$.

119. To reduce an improper fraction to a whole or mixed quantity.

Divide the numerator by the denominator, as in Art. 103.

21. Thus $\frac{ab+bm+d}{b}=a+m+\frac{d}{b}$.

22. Reduce $\frac{am-a+ady-hr}{a}$, to a mixed quantity.

120. To reduce a mixed quantity to an improper fraction.

Multiply the integer by the given denominator, and add the given numerator to the product. (Art. 113.) *The sum will be the required numerator; and this placed over the given denominator will form the improper fraction required.*

N. B. If the sign before the dividing line is —, all the signs in the numerator must be changed. (Art. 114, note.)

Quest.—How reduce an improper fraction to a whole or mixed quantity? How reduce a mixed quantity to an improper fraction? When the sign — is before the dividing line, what must be done?

23. Reduce $a+\frac{1}{b}$ to an improper fraction. Ans. $\frac{ab+1}{b}$

24. Reduce $a-\frac{b}{c}$. 25. Reduce $x-\frac{a+b}{c}$.

26. Reduce $ab-\frac{a-c}{x}$. 27. Reduce $ax+\frac{a-b}{d}$.

28. Reduce $m+d-\frac{r}{h-d}$. 29. Reduce $b-\frac{c}{d-y}$.

121. To reduce a compound fraction to a simple one.

Multiply all the numerators together for a new numerator, and all the denominators for a new denominator.

30. Reduce $\frac{2}{7}$ of $\frac{a}{b+2}$. Ans. $\frac{2a}{7b+14}$.

31. Reduce $\frac{2}{3}$ of $\frac{4}{5}$ of $\frac{b+h}{2a-m}$. 32. Reduce $\frac{1}{7}$ of $\frac{1}{3}$ of $\frac{1}{8-a}$. ×

EXAMPLES FOR PRACTICE.

1. What is the value of $\frac{8axy}{2axy}$?

2. What is the value of $\frac{aabbccddff}{abcdf}$?

3. What is the value of $\frac{ab}{a}\times 4$?

4. What is the value of $\frac{16axy}{a}\div 4x$?

5. What is the value of $\frac{16ax}{2a}$ when the denom. is $\times 4$?

6. What is the value of $\frac{3axy}{24ax}$ when the denom. is $\div 6ax$?

QUEST. How reduce a compound fraction to a simple one?

7. What is the value of $\frac{17abx}{34a}$ when both numerator and denominator are $\times 2d$?

8. Reduce $\frac{6abc+12abx}{2ab}$ to a whole or mixed number.

9. Reduce $-\frac{24xy-48ax}{12x}$ to a whole or mixed number.

10. Reduce $\frac{ab+e+dx+ax+am}{a}$ to a whole or mixed No.

Reduce the four next examples to the lowest terms.

11. $\frac{abc}{aac}$. 12. $\frac{3xy}{12xyy}$. 13. $\frac{bx+by}{ab+bx}$. 14. $\frac{aaxy-aab}{ac+abc}$.

15. Reduce $\frac{ax}{y}$ and $\frac{c}{d}$ to a common denominator.

16. Reduce $\frac{a}{b}$; $\frac{c}{d}$; $\frac{f}{g}$ and $\frac{x}{y}$ to a common denominator.

17. Reduce $a-\frac{b+c}{x}$ to an improper fraction.

18. Reduce $a+b-\frac{x-y}{4m}$ to an improper fraction.

19. Reduce $\frac{2}{3}$ of $\frac{a}{b}$ of $\frac{c}{d}$ of $\frac{x}{y}$ to a simple fraction.

20. Reduce $\frac{2}{7}$ of $\frac{2x}{4b}$ of $\frac{4ab}{2}$ of $\frac{2c}{4x}$ of $\frac{4dx}{2a}$ of $\frac{abc}{2d}$ to a simple fraction.

ADDITION OF FRACTIONS.

122. Rule.—*Reduce the fractions to a common denominator; then add their numerators, and place the sum over the common denominator.*

Quest.—How are fractions added?

EXAMPLES.

1. Add $\frac{2}{16}$ and $\frac{4}{16}$ of a pound. Ans. $\frac{2+4}{16}$ or $\frac{6}{16}$.

2. Add $\frac{a}{b}$ and $\frac{c}{d}$. First reduce them to a common denominator. They will then be $\frac{ad}{bd}$ and $\frac{bc}{bd}$ and their sum $\frac{ad+bc}{bd}$.

3. Given $\frac{m}{d}$ and $-\frac{2r+d}{3h}$, to find their sum.

4. Given $\frac{a}{d}$ and $-\frac{b-m}{y}$

5. Given $\frac{a}{y}$ and $\frac{c}{-m}$.

6. Given $\frac{a}{a+b}$ and $\frac{b}{a-b}$.

7. Add $\frac{-a}{d}$ to $\frac{-h}{m-r}$

8. Add $\frac{-4}{2}$ to $\frac{-16}{7-3}$.

9. Add $\frac{4a}{b}$, $\frac{6c}{d}$ to $\frac{3m}{3x}$

10. Add $\frac{2xy}{2}$, $\frac{hx}{y}$, to $\frac{ax}{c}+\frac{2}{a}$.

11. Add $a+\frac{b}{2}$, $c+\frac{d}{x}$, xy and $\frac{a-b}{4}$.

12. Add $42-\frac{2b}{c}$, $d-\frac{b+c}{3x}$ and $a+\frac{b+c}{3x}$.

13. Add $\frac{3a}{2c}-\frac{x-y}{c}$, $\frac{a}{2c}$, $\frac{xyy}{xy}$ and $\frac{8ab}{4c}$.

14. Add $2a+x$, $\frac{8x+10}{2}$ and $-\frac{3bx+4b}{b}$

123. For many purposes, it is sufficient to add fractions in the same manner as integers are added, by writing them one after another with their signs. (Art. 47.)

QUEST.—What other way?

15. Thus the sum of $\frac{a}{b}$ and $\frac{3}{y}$ and $-\frac{d}{2m}$, is $\frac{a}{b}+\frac{3}{y}-\frac{d}{2m}$.

124. To add fractions and integers.

Write them one after another with their signs; or convert the integer into a fraction, (Art. 113), *and then add their numerators.*

16. What is the sum of a and $\frac{b}{m}$?

17. What is the sum of $3d$ and $\frac{h+d}{m-y}$?

18. What is the sum of $5x$ and $\frac{a+3b}{c}$?

SUBTRACTION OF FRACTIONS.

125. Rule.—*Change the signs of the fractions to be subtracted from $+$ to $-$, and from $-$ to $+$; and then proceed as in addition of fractions.* (Art. 122.)

1. From $\frac{a}{b}$ subtract $\frac{h}{m}$.

First, Reduce the fractions to a *common denominator.*

Thus, $\begin{cases} a\times m=am, \text{ the numerator of the minuend.} \\ h\times b=bh, \text{ the numerator of the subtrahend.} \\ b\times m=bm, \text{ the common denominator.} \end{cases}$

The fractions become $\frac{am}{bm}$ and $\frac{bh}{bm}$.

Second, Change the sign before the *dividing line* of the subtrahend, as $\frac{am}{bm}-\frac{bh}{bm}$.

Quest.—How are integers and fractions added? What is the rule for subtraction of fractions? What sign do you change?

Third, *Unite the terms* as in addition of fractions; thus, $\frac{am}{bm} - \frac{bh}{bm} = \frac{am-bh}{bm}$. Ans.

2. From $\frac{a+y}{r}$ subtract $\frac{h}{d}$.

3. From $\frac{a}{m}$ subtract $\frac{d-b}{y}$.

4. From $\frac{a+3d}{4}$ subtract $\frac{3a+2d}{3}$.

5. From $\frac{b-d}{m}$ subtract $\frac{b}{y}$.

6. From $\frac{a+1}{d}$ subtract $\frac{d-1}{m}$.

7. From $\frac{3}{a}$ subtract $\frac{4}{b}$.

126. Fractions may also be subtracted, like integers, by setting them down, after their signs are changed, without reducing them to a common denominator.

8. From $\frac{h}{m}$ subtract $-\frac{h+d}{y}$. Ans. $\frac{h}{m} + \frac{h+d}{y}$.

127. To subtract an integer from a fraction, or a fraction from an integer.

Change the sign of the subtrahend, and write it after the minuend; or, throw the integer into the form of a fraction, (Art. 113), *and then proceed according to the general rule for subtraction of fractions.*

10. From $\frac{h}{y}$ subtract m. Ans. $\frac{h}{y} - m = \frac{h-my}{y}$.

Quest.—How subtract an integer from a fraction, or a fraction from an integer?

11. From $4a+\frac{b}{c}$ subtract $3a-\frac{h}{d}$.

12. From $1+\frac{b-c}{d}$ subtract $\frac{c-b}{d}$.

13. From $a+3h-\frac{d-b}{2}$ subtract $3a-h+\frac{d+b}{3}$.

14. From $\frac{a-x}{b}$ take $\frac{d+y}{c}$. 15. From $\frac{a+b}{x}$ take $\frac{c-d}{y}$.

16. From $\frac{a}{b-x}$ take $\frac{c}{d+y}$. 17. From $a-\frac{x}{y}$ take $\frac{3d}{2}$.

18. From $x+y$ take $\frac{a-b}{c}$. 19. From $\frac{x-y}{10}$ take $\frac{a-b}{x+y}$.

20. From $x-\frac{4y-2c}{2}$ take $\frac{3x-6y}{3}+a$.

MULTIPLICATION OF FRACTIONS.

128. By the definition of multiplication, multiplying by a *fraction* is taking a *part* of the *multiplicand* as many times as there are *like parts* of an *unit* in the *multiplier*. (Art. 71.)

Suppose a is to be multiplied by $\frac{3}{4}$.

A fourth part of a is $\frac{a}{4}$.

This taken 3 times is $\frac{a}{4}+\frac{a}{4}+\frac{a}{4}=\frac{3a}{4}$.

130. Hence, to multiply a fraction by a fraction,

Multiply the numerators together for a new numerator, and the denominators together for a new denominator.

QUEST.—What is meant by multiplying by a fraction? Rule to multiply a fraction by a fraction?

1. Multiply $\frac{3b}{c}$ into $\frac{d}{2m}$. Ans. $\frac{3bd}{2cm}$.

2. Multiply $\frac{a+d}{y}$ into $\frac{4h}{m-2}$.

3. Multiply $\frac{(a+m)\times h}{3}$ into $\frac{4}{(a-n)}$.

4. Multiply $\frac{a+h}{3+d}$ into $\frac{4-m}{c+y}$.

5. Multiply $\frac{1}{a+3r}$ into $\frac{3}{8}$.

6. Multiply together $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{m}{y}$.

7. Multiply $\frac{2a}{m}$, $\frac{h-d}{y}$, $\frac{b}{c}$ and $\frac{1}{r-1}$.

8. Multiply $\frac{3+b}{n}$, $\frac{1}{h}$ and $\frac{d}{r+2}$.

9. Multiply $\frac{ad}{hy}$, $\frac{a-6}{d+1}$ and $\frac{3}{7}$.

131. *The multiplication may often be shortened, by rejecting equal factors from the numerators and denominators.*

10. Multiply $\frac{a}{r}$ into $\frac{h}{a}$ and $\frac{d}{y}$. Ans. $\frac{dh}{ry}$.

Here a, being in one of the numerators, and in one of the denominators, may be omitted. If it be retained, the product will be $\frac{adh}{ary}$. But this reduced to lower terms, by Art. 117, will become $\frac{dh}{ry}$ as before.

Quest.—How shorten the process, when the numerators and denominators contain equal factors?

11. Multiply $\frac{ad}{m}$ into $\frac{m}{3a}$ and $\frac{ah}{2d}$.

12. Multiply $\frac{a+d}{y}$ into $\frac{my}{ah}$.

13. Multiply $\frac{am+d}{h}$ into $\frac{h}{m}$ and $\frac{3r}{5a}$.

132. To multiply a fraction and an integer together.

Either multiply the numerator of the fraction by the integer; or divide the denominator by the integer.

14. Thus $\frac{m}{y}\times a$ is $\frac{am}{y}$. For $a=\frac{a}{1}$; and $\frac{a}{1}\times\frac{m}{y}=\frac{am}{y}$. Ans.

15. Multiply $\frac{m}{ax}$ into a. Dividing the denominator by a, we have $\frac{m}{x}$.

And multiplying the numerator by a, we have $\frac{am}{ax}$. But $\frac{am}{ax}=\frac{m}{x}$, which is the same result.

133. *A fraction is multiplied into a quantity equal to its denominator, by cancelling the denominator.* Thus,

16. $\frac{a}{b}\times b=a$. For $\frac{a}{b}\times b=\frac{ab}{b}$. But the letter b, being in both the numerator and denominator, may be set aside.

17. Mult. $\frac{3m}{a-y}$ into $(a-y)$. 18. Mult. $\frac{h+3d}{3+m}$ into $(3+m)$.

N. B. On the same principle, a fraction is multiplied into any *factor* in its *denominator*, by *cancelling that factor.*

19. Mult. $\frac{a}{by}$ into y. 20. Mult. $\frac{h}{24}$ into 6.

QUEST.—How multiply a fraction and an integer? How by a quantity equal to its denominator?

EXAMPLES FOR PRACTICE.

1. Multiply $\frac{a}{2}$, $\frac{4a}{5}$, and $\frac{10a}{21}$.

2. Mult. $\frac{x}{a}$ into $\frac{a+x}{a-x}$.

3. Mult. $\frac{3x}{2}$ into $\frac{5x}{3b}$.

4. Mult. $\frac{3x+y}{24a+32c}$ into 8.

5. Mult. $\frac{a+b}{20x+25xy}$ into $5x$.

6. Mult. $\frac{3}{x}$ into $\frac{4x}{3}$ into $\frac{a}{b}$.

7. Mult. $\frac{abcd}{3x+y}$ into $\frac{3x+y}{abcd}$.

8. Mult. $\frac{a-b}{x}$ into $\frac{y}{a-b}$.

9. Mult. $\frac{a}{b}\times\frac{c}{d}\times\frac{3}{4}\times\frac{b}{a}$.

10. Mult. $\frac{a+b}{4}$ into $\frac{a-b}{3}$.

11. Mult. $a\times\frac{bc}{3x}\times 6x$.

12. Mult. $\frac{24ab}{3x}\times\frac{3xy}{8a}\times\frac{3}{4}$.

DIVISION OF FRACTIONS.

134. To divide a fraction by a fraction.

Invert the divisor, and then proceed as in multiplication of fractions. (Art. 130.)

1. Divide $\frac{a}{b}$ by $\frac{c}{d}$. Ans. $\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}$.

To understand the reason of the rule, let it be premised, that the product of any fraction into the same fraction inverted, is always a unit.

Thus $\frac{a}{b}\times\frac{b}{a}=\frac{ab}{ab}=1$. And $\frac{d}{h\times y}\times\frac{h+y}{d}=1$.

But a quantity is not altered by multiplying it by a unit. Therefore, if a dividend be multiplied, first into the divisor inverted, and then into the divisor itself, the last product will

QUEST.—How divide one fraction by another? Explain the reason of the rule?

be equal to the dividend. Now, by the definition, (Art. 91,) "division is finding a quotient, which multiplied into the divisor will produce the dividend." And as the dividend multiplied into the divisor inverted is such a quantity, the quotient is truly found by the rule.

2. Divide $\frac{m}{2d}$ by $\frac{3h}{y}$. Ans. $\frac{m}{2d} \times \frac{y}{3h} = \frac{my}{6dh}$

Proof. $\frac{my}{6dh} \times \frac{3h}{y} = \frac{m}{2d}$, the dividend.

3. Divide $\frac{x+d}{r}$ by $\frac{5d}{h}$. 4. Divide $\frac{4dh}{x}$ by $\frac{4hr}{a}$.

5. Divide $\frac{36d}{5}$ by $\frac{18h}{10y}$. 6. Divide $\frac{ab+1}{3y}$ by $\frac{ab-1}{x}$.

7. Divide $\frac{h-my}{4}$ by $\frac{3}{a+1}$.

135. To divide a fraction by an integer.

Divide the numerator by the given integer, when it can be done without a remainder; but when this cannot be done, multiply the denominator by the integer.

8. Thus the quotient of $\frac{am}{b}$ divided by m, is $\frac{a}{b}$.

9. Div. $\frac{1}{a-b}$ by h. Ans. $\frac{1}{ah-bh}$. 10. Div. $\frac{3}{4}$ by 6

136. To divide an integer by a fraction

Reduce the integer to the form of a fraction, (Art. 113,) *and proceed as in* Art. 134. Or, *multiply the integer by the denominator, and divide the product by the numerator.*

QUEST.—How many ways to divide a fraction by an integer? What are they? How does it appear, that multiplying the denominator divides a fraction? How divide an integer by a fraction?

11. Div. a by $\frac{c}{d}=\frac{a}{1}\div\frac{c}{d}=\frac{ad}{c}$. Or, $a\div\frac{c}{d}=\frac{a\times d}{c}=\frac{ad}{c}$.

12. Div. xy by $\frac{a+b}{2}$. 13. Div. $ab+cx$ by $\frac{3am}{12d}$.

14. Div. $3ac-x$ by $\frac{a}{3}$.

136.*a*. By the definition, (Art. 32,) "the *reciprocal* of a quantity is the quotient arising from dividing a unit by that quantity."

Therefore the reciprocal of $\frac{a}{b}$ is $1\div\frac{a}{b}=1\times\frac{b}{a}=\frac{b}{a}$. That is,

The reciprocal of a fraction is the fraction inverted.

Thus, the reciprocal of $\frac{b}{m+y}$ is $\frac{m+y}{b}$; the reciprocal of $\frac{1}{3y}$ is $\frac{3y}{1}$ or $3y$; the reciprocal of $\frac{1}{4}$ is 4.

Hence the reciprocal of a fraction whose numerator is 1, is the *denominator* of the fraction.

Thus, the reciprocal of $\frac{1}{a}$ is a; of $\frac{1}{a+b}$, is $a+b$, &c.

EXAMPLES FOR PRACTICE.

1. Divide $\frac{3abc}{x-y}$ by $3ab$.
2. Divide $\frac{10axx+15abx}{10-y}$ by $5ax$.
3. Divide $\frac{3x+11}{x}$ by $3a$.

QUEST.—What is the reciprocal of a fraction?

4. Divide $\frac{a+1-x}{2cd}$ by d.

5. Divide $\frac{a+b}{5}$ by $\frac{a}{b}$.

6. Divide $\frac{x}{3ab}$ by $\frac{y}{4+2x}$.

7. Divide $\frac{a+b}{3}$ by $\frac{4}{a-b}$.

8. Divide $\frac{x+y}{a}$ by $\frac{b}{a}$.

9. Divide $\frac{3ab-6xy}{6}$ by $\frac{ab-2xy}{2}$.

10. Divide $21abc$ by $\frac{7ab}{x}$.

11. Divide $8xy$ by $\frac{2ab}{c}$.

12. Divide $18ax$ by $\frac{2a(x-y)}{3m}$.

13. Divide $\frac{18(a+x)}{3}$ by $\frac{2a(a-y)}{2m}$.

14. Divide $2a+\frac{3c+d}{x}$ by $x+\frac{y+b}{2}$.

SECTION VII.

SIMPLE EQUATIONS.

137. MOST of the investigations in algebra are carried on by means of *equations*. In the solution of problems, for example, we represent the unknown quantity, or number sought

by a certain letter; and then, to ascertain the *value* of this unknown quantity, or letter, we form an *algebraic expression* from the conditions of the question, which is *equal* to some given quantity or number. Thus,

A drover bought an equal number of sheep and cows for \$840. He paid \$2 a head for the sheep, and \$12 a head for the cows. How many did he buy of each?

OPERATION.

Let x=the number bought of each.
then $2x$=the price of all the sheep,
and $12x$= " " cows. Hence,
$2x+12x=840$ by the conditions. (Ax. 9.)
$14x=840$ by uniting the x's.
and $x=60$, the number bought of each.

It will be perceived that the *unknown quantity* or *number sought*, is represented by the letter x; and from the conditions of the problem we obtain the quantity $14x$, which is equal to the given quantity \$840. This whole *algebraic* expression, $14x=\$840$, is called an *equation*. Hence,

138. *An* EQUATION *is a proposition expressing in algebraic characters the equality between one set of quantities and another, or between different expressions for the same quantity.*

This *equality* is denoted by the sign $=$, which is read "is equal to," or "equals." Thus, $x+a=b+c$; and $5+8=17-4$, are equations in which the sum of x and a is equal to the sum of b and c; and the sum of 5 and 8 is equal to the difference of 17 and 4.

QUEST.—How are investigations generally carried on in Algebra? What is an equation? What are the members of an equation?

The quantities on the two sides of the sign = are called *members* of the equation; the several terms on the *left* constituting the *first* member, and those on the *right* the *second* member.

139. When the unknown quantity is of the *first power*, as $3x$, the proposition is called a *simple equation;* or an equation of the *first degree.*

140. *The reduction of an equation consists, in bringing the unknown quantity by itself on one side, and all the known quantities on the other side, without destroying the equality of the members.*

To effect this, it is evident that one of the members must be as much increased or diminished as the other. If a quantity be added to one, and not to the other, the equality will be destroyed. But the members will remain equal,

If the same or equal quantities be *added* to each. Ax. 1.
If the same or equal quantities be *subtracted* from each. Ax. 2.
If each be *multiplied* by the same or equal quantities. Ax. 3.
If each be *divided* by the same or equal quantities. Ax. 4.

140.*a*. The *principal* reductions in simple equations are those which are effected by *transposition*, *multiplication* and *division*.

REDUCTION OF EQUATIONS BY TRANSPOSITION.

141. In the equation $x-7=9$, the number 7 being connected with the unknown quantity x by the sign $-$, the one is *subtracted* from the other. To reduce the equation, let 7 be *added* to both sides. It then becomes $x-7+7=9+7$. (Art. 59.)

The equality of the members is preserved, because one is increased as much as the other. (Axiom 1.) But on one

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QUEST.—What is a simple equation? In what does the reduction of an equation consist? How are the principal reductions effected?

side, we have -7 and $+7$. As these are equal, and have contrary signs, they *balance each other*, and may be cancelled. The equation will then be $x=9+7$. (Art. 54.)

Here the value of x is found. It is shown to be equal to $9+7$, that is to 16. The equation is therefore reduced. The unknown quantity is on one side by itself, and all the known quantities on the other side.

In the same manner, if $x-b=a$
Adding b to both sides $x-b+b=a+b$
And cancelling $(-b+b)$ $x=a+b$. Hence,

142. *When known quantities are connected with the unknown quantity by the sign $+$ or $-$, the equation is reduced by* TRANSPOSING *the known quantities to the other side, and prefixing the contrary sign.*

This is called reducing an equation by *addition* or *subtraction*, because it is, in effect, adding or subtracting certain quantities, to or from each of the members.

1. Reduce the equation $x+3b-m=h-d$
Transposing $+3b$ we have $x-m=h-d-3b$
And transposing $-m$, $x=h-d-3b+m$.

143. When several terms on the same side of an equation are *alike*, they must be united in one, by the rules for reduction in addition. (Arts. 50, 51.)

2. Reduce the equation $x+5b-4h=7b$
Transposing $5b-4h$ $x=7b-5b+4h$
Uniting $7b-5b$ in one term $x=2b+4h$.

144. The *unknown* quantity must also be transposed, whenever it is on both sides of the equation. It is not material on which side it is finally placed.

QUEST.—Rule to reduce an equation by transposition? How does it appear that this does not destroy the equality of the members? When several terms are alike, what must be done? When the unknown quantity is on both sides of the equation, what?

3. Reduce the equation $2x+2h=h+d+3x$

By transposition, $2h-h-d=3x-2x$

And $h-d=x$.

145. When the *same term*, with the same sign, is on *opposite sides* of the equation, instead of transposing, we may *expunge* it from each. For this is only subtracting the same quantity from equal quantities. (Ax. 2.)

4. Reduce the equation $x+3h+d=b+3h+7d$

Expunging $3h$ $x+d=b+7d$

And $x=b+6d$.

146. As *all* the terms of an equation may be transposed, or supposed to be transposed, and it is immaterial which member is written first, it is evident that the *signs of all the terms may be changed*, without affecting the equality.

Thus, if we have $x-b=d-a$

Then by transposition, $-d+a=-x+b$

Or, inverting the members, $-x+b=-d+a$.

147. If all the terms on *one* side of an equation be transposed, each member will be equal to 0.

Thus, if $x+b=d$, then $x+b-d=0$.

5. Reduce $a+2x-8=b-4+x+a$.

6. Reduce $y+ab-hm=a+2y-ab+hm$.

7. Reduce $h+30+7x=8-6h+6x-d+b$.

8. Reduce $bh+21-4x+d=12-3x+d-7bh$.

9. Reduce $5x+10+a=25+4x+a$.

10. Reduce $5c+2x+12-3=x+20+5c$.

11. Reduce $a+b-3x=20+a-4x+b$.

12. Reduce $x+3-2x-4=34+3x-4-5x$.

QUEST.—When the same term with the same sign is on opposite sides, what? What is the effect when all the signs of both members are changed at the same time? If all the terms on one side are transposed to the other, to what is each member equal?

REDUCTION OF EQUATIONS BY MULTIPLICATION.

148. The unknown quantity, instead of being connected with a known quantity by the sign $+$ or $-$, may be *divided* by it, as in the equation $\frac{x}{a}=b$.

Here the reduction can not be made, as in the preceding instances, by transposition. But if both members be *multiplied* by a, the equation will become, $x=ab$. (Art. 140.)

For a fraction is multiplied into its denominator, by removing the denominator. (Art. 133.) Hence,

149. *When the unknown quantity is* DIVIDED *by a known quantity, the equation is reduced by* MULTIPLYING *every term on each side by this known quantity.* (Ax. 3.)

N. B. The same transpositions are to be made in this case, as in the preceding examples.

13. Reduce the equation $\frac{x}{c}+a=b+d$

Multiplying both sides by c

The product is $x+ac=bc+cd$

And $x=bc+cd-ac$.

14. Reduce the equation $\frac{x-4}{6}+5=20$.

15. Reduce the equation $\frac{x}{a+b}+d=h$.

150. When the unknown quantity is in the *denominator* of a fraction, the reduction is made in a similar manner, by multiplying the equation by this denominator.

QUEST.—How is an equation reduced by multiplication? How is a fraction multiplied into its denominator? How does it appear that this method of reducing equations does not destroy the equality? When the unknown quantity is in the denominator, how proceed?

16. Reduce the equation $\frac{6}{10-x}+7=8$.

151. Though it is not generally *necessary*, yet it is often convenient, to remove the denominator from a fraction consisting of *known* quantities only. This may be done in the same manner as the denominator is removed from a fraction, which contains the unknown quantity.

17. Take for example $\frac{x}{a}=\frac{d}{b}+\frac{h}{c}$

Multiplying by a $\quad x=\frac{ad}{b}+\frac{ah}{c}$

Multiplying by b $\quad bx=ad+\frac{abh}{c}$

Multiplying by c $\quad bcx=acd+abh$.

152. *An equation may be cleared of* FRACTIONS *by multiplying each side into all the* DENOMINATORS.

Obser. In clearing an equation of fractions, it often happens, that a numerator becomes a *multiple* of its denominator, (i. e. can be divided by it without a remainder,) or that some of the fractions can be reduced to *lower* terms. When this occurs, the operation *may be shortened* by performing the division, and reducing the fractions to the lowest terms according to Art. 117.

18. Reduce the equation $\frac{x}{a}=\frac{b}{d}+\frac{e}{g}-\frac{h}{m}$.

19. Reduce the equation $\frac{x}{2}=\frac{2}{3}+\frac{4}{5}+\frac{6}{2}$.

153. N. B. In clearing an equation of fractions, it will be necessary to observe, that the sign — prefixed to any frac-

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QUEST.—How clear an equation of fractions? How prove that this does not destroy the equality? When a numerator becomes a multiple of its denominator, what may be done? When a fraction can be reduced to lower terms, what? What must be observed as to the sign — before the dividing line?

tion, denotes that the whole value is to be subtracted, which is done by changing the signs of all the terms in the numerator. (Art. 114.)

20. Reduce $\frac{a-d}{x}=c-\frac{3b-2hm-6n}{r}$.

21. Reduce $\frac{x}{3}-\frac{x}{4}=6$.

22. Reduce $\frac{4x}{5}=\frac{3}{5}+\frac{3x}{5}+\frac{8}{10}$.

23. Reduce $2x-\frac{9x}{5}=\frac{10}{25}+\frac{8}{5}$.

24. Reduce $-x+\frac{x}{2}+\frac{3x}{4}-\frac{2x}{7}+\frac{x}{14}=\frac{10}{4}$.

REDUCTION OF EQUATIONS BY DIVISION.

154. *When the unknown quantity is* MULTIPLIED *into any known quantity, the equation is reduced by* DIVIDING *every term on both sides by this known quantity.* (Ax. 4.)

25. Reduce the equation $ax+b-3h=d$
By transposition $ax=d+3h-b$
Dividing by a $x=\frac{d+3h-b}{a}$.

26. Reduce the equation $2x=\frac{a}{c}-\frac{d}{h}+4b$.

155. If the unknown quantity has co-efficients in *several terms*, the equation must be divided by *all* these co-efficients, connected by their signs, according to Art. 98.

QUEST.—What does this sign, when thus situated, show? When the unknown quantity has a co-efficient, how reduce the equation? If the unknown quantity has a co-efficient in several terms, how?

27. Reduce the equation $3x-bx=a-d$

That is, (Art. 97,) $(3-b)\times x=a-d$.

Dividing by $3-b$ $x=\frac{a-d}{3-b}$. Ans.

28. Reduce the equation $ax+x=h-4$.

29. Reduce the equation $x-\frac{x-b}{h}=\frac{a+d}{4}$.

156. If any quantity, either known or unknown, is found as a factor in *every term*, the equation may be *divided* by it. On the other hand, if any quantity is a *divisor* in every term, the equation may be *multiplied* by it. In this way, the factor or divisor will be removed, so as to render the expression more simple.

30. Reduce the equation $ax+3ab=6ad+a$

Dividing by a $x+3b=6d+1$

And $x=6d+1-3b$.

31. Reduce the equation $\frac{x+1}{x}-\frac{b}{x}=\frac{h-d}{x}$.

Multiplying by x, (Art. 133,) $x+1-b=h-d$

And $x=h-d+b-1$.

32. Reduce the equation $x\times(a+b)-a-b=d\times(a+b)$.

157. *A proportion is converted into an equation by making the product of the extremes, one side of the equation; and the product of the means, the other side.*

33. Reduce to an equation $ax:b::ch:d$.

The product of the extremes is adx

The product of the means is bch

The equation is, therefore, $adx=bch$.

34. Reduce to an equation $a+b:c::h-m:y$.

QUEST.—If any quantity is found as a factor in every term, how? How convert a proportion into an equation?

158. *On the other hand, an equation may be converted into a proportion, by resolving one side of the equation into two factors, for the middle terms of the proportion; and the other side into two factors, for the extremes.*

35. Convert the equation, $adx=bch$, into a proportion. The first member may be divided into the two factors ax and d; the second into ch and b. From these factors we may form the proportion $ax:b::ch:d$.

36. Reduce to a proportion $ay+by=ch-cm$.

37. Reduce the equation $16x+2=34$.

38. Reduce the equation $4x-8=-3x+13$.

39. Reduce the equation $10x-19=7x+17$.

40. Reduce the equation $8x-3+9=-7x+9+27$.

SUBSTITUTION.

159. In the reduction of an equation, as well as in other parts of algebra, a *complicated* process can often be rendered *shorter and more simple*, by using letters for the given numbers when large, (Art. 35;) and also by introducing a *new letter* which shall be made to represent a *whole algebraic expression.*

160. *This process is called* SUBSTITUTION. After the operation is completed, the *numbers*, or the *compound quantity* for which a *single* letter has been *substituted*, must be restored.

41. Reduce $\frac{x}{750}+\frac{3}{375}=1$.

Clearing of fractions, $375x+3\times750=1\times750\times375$

and $x=\frac{281250-2250}{375}=744$. Ans.

QUEST.—How can an equation be converted into a proportion? What is meant by substitution? What is the advantage of it? After the operation is performed, what must be done?

By substituting a for 750; b for 3; and c for 375; the equation becomes $\frac{x}{a}+\frac{b}{c}=1$.

Clearing of fractions, $cx+ab=ac$: and $x=a-\frac{ab}{c}$.

Restoring the numbers, $x=750-\frac{3\times 750}{375}=744$. Ans.

42. Reduce $\frac{3x}{4}+6=84$. Substitute a for 3; b for 4; c for 6; and d for 84.

43. Reduce $\frac{x}{350}+\frac{4500}{7000}=10$. Substitute a for 350; b for 4500; c for 7000; and d for 10.

44. Reduce $\frac{x}{m+n}+\frac{a}{c}=b$. Substitute d for $(m+n)$, and the equation is $\frac{x}{d}+\frac{a}{c}=b$.

Clearing of fractions, $cx+ad=bcd$; and $x=\frac{bcd-ad}{c}$: restoring $(m+n)$; $x=\frac{bc(m+n)-a(m+n)}{c}$.

45. Reduce $\frac{x}{l-m-n}+\frac{d}{c}=ab$. Substitute h for $(l-m-n)$.

46. Reduce $\frac{x}{m}-\frac{a}{b+c+d}=cd$. Substitute h for $(b+c+d)$.

EXAMPLES FOR PRACTICE.

1. Reduce $\frac{3x}{4}+6=\frac{5x}{8}+7$.

2. Reduce $\frac{x}{a}+h=\frac{x}{b}-\frac{x}{c}+d$.

3. Reduce $40-6x-16=120-14x$.

4. Reduce $\frac{x-3}{2}+\frac{x}{3}=20-\frac{x-19}{2}$.

5. Reduce $\frac{x}{3}+\frac{x}{5}=20-\frac{x}{4}$.

6. Reduce $\frac{1-a}{x}-4=5$.

7. Reduce $\frac{3}{x+4}-2=8$.

8. Reduce $\frac{6x}{x+4}=1$.

9. Reduce $x+\frac{x}{2}+\frac{x}{3}=11$.

10. Reduce $\frac{x}{2}+\frac{x}{3}-\frac{x}{4}=\frac{7}{10}$.

11. Reduce $\frac{x-5}{4}+6x=\frac{284-x}{5}$.

12. Reduce $3x+\frac{2x+6}{5}=5+\frac{11x-37}{2}$

13. Reduce $\frac{6x-4}{3}-2=\frac{18-4x}{3}+x$.

14. Reduce $21+\frac{3x-11}{16}=\frac{5x-5}{8}+\frac{97-7x}{2}$

15. Reduce $3x-\frac{x-4}{4}-4=\frac{5x+14}{3}-\frac{1}{12}$.

16. Reduce $\frac{7x+5}{3}-\frac{16+4x}{5}+6=\frac{3x+9}{2}$.

17. Reduce $\frac{17-3x}{5}-\frac{4x+2}{3}=5-6x+\frac{7x+14}{3}$.

18. Reduce $x-\frac{3x-3}{5}+4=\frac{20-x}{2}-\frac{6x-8}{7}+\frac{4x-4}{5}$.

19. Reduce $\frac{6x+7}{9}+\frac{7x-13}{6x+3}=\frac{2x+4}{3}$.

20. Reduce $\frac{5x+4}{2}:\frac{18-x}{4}::7:4$.

SOLUTION OF PROBLEMS.

161. For the solution of problems in Simple Equations, we derive from the preceding principles the following

GENERAL RULE.

I. *Translate the statement of the question from common to algebraic language, in such a manner as to form an equation,* i. e. put the question into an equation. (Art. 33.)

II. *Clear the equation of fractions by multiplying every term on each side by all the denominators.* (Art. 152.)

III. *Transpose all the terms containing the unknown quantity to one side, and all the known quantities to the other, taking care to change the signs of the terms transposed, and unite the terms that are alike.* (Art. 50, 51.)

IV. *Remove the co-efficients of the unknown quantity, by dividing all the terms in the equation by them.* (Art. 154.)

Proof.—*Substitute the value of the unknown quantity for the letter itself in the equation; and if the number satisfies the conditions of the question, it is the answer sought.*

Problem 1. A man being asked how much he gave for his watch, replied; If you multiply the price by 4, and to the product add 70, and from this sum subtract 50, the remainder will be equal to 220 dollars.

To solve this question, we must first translate the conditions of the problem into such algebraic expressions as will form an *equation*.

Quest.—What is the first step in the solution of a problem? Second? Third? Fourth? Proof?

Let x=the price of the watch.

This price is to be mult'd by 4, which makes $4x$

To the product, 70 is to be added, making $4x+70$

From this, 50 is to be subtracted, making $4x+70-50$.

Here we have a number of the conditions, expressed in algebraic terms; but have as yet no *equation*. We must observe then, that by the last condition of the problem, the preceding terms are said to be *equal* to 220.

We have, therefore, this equation $4x+70-50=220$

Which reduced gives $x=50$

Here the value of x is found to be 50 dollars, which is the price of the watch.

Proof.—The original equation is $4x+70-50=220$

Substituting 50 for x, it becomes $4\times50+70-50=220$

That is, $220=220$.

Prob. 2. What number is that, to which, if its half be added, and from the sum 20 be subtracted, the remainder will be a fourth of the number itself?

In stating questions of this kind, where fractions are concerned, it should be recollected, that $\frac{1}{3}x$ is the same as $\frac{x}{3}$; that $\frac{2}{5}x=\frac{2x}{5}$, &c. (Art. 108.)

Let x=the number required.

Then, by the conditions proposed, $x+\frac{x}{2}-20=\frac{x}{4}$

And reducing the equation $x=16$.

Proof, $16+\frac{16}{2}-20=\frac{16}{4}$.

Prob. 3. A father divides his estate among his three sons, in such a manner, that,

The first has \$1000 less than half of the whole;

The second has 800 less than one-third of the whole;

The third has 600 less than one fourth of the whole;
What is the value of the estate?

Prob. 4. Divide 48 into two such parts, that if the less be divided by 4, and the greater by 6, the sum of the quotients will be 9.

Let $x =$ the smaller part.
Then $48-x =$ the greater part.

By the conditions of the problem, $\frac{x}{4}+\frac{48-x}{6}=9.$

162. Letters may be employed to express the *known* quantities in an equation, as well as the unknown. (Art. 159.) A particular value is assigned to the letters, when they are introduced into the calculation; and at the close, the numbers are restored. (Art. 35.)

Prob. 5. If to a certain number, 720 be added, and the sum be divided by 125, the quotient will be equal to 7392 divided by 462. What is the number?

Let $x=$ the number required.

$a=720 \qquad d=7392$

$b=125 \qquad h=462$

Then by the conditions of the problem, $\frac{x+a}{b}=\frac{d}{h}.$

Therefore $$x=\frac{bd-ah}{h}.$$

Restoring the numbers, $x=\frac{(125\times7392)-(720\times462)}{462}=1280.$

162.*a*. When the solution of an equation brings out a *negative* answer, it shows that the *value* of the unknown quantity

QUEST.—When letters are substituted for known quantities, what must be done at the close of the calculation? When the solution brings out a negative answer, what does it show?

is *contrary* to the quantities, which in the statement of the question are considered *positive*. But this being determined by the answer, the omission of the sign $-$ before the unknown quantity in the course of the calculation, can lead to no mistake.

Prob. 6. A merchant gains or loses, in a bargain, a certain sum. In a second bargain, he gains 350 dollars, and in a third, he loses sixty. In the end he finds he has gained 200 dollars, by the three together. How much did he gain or lose by the first?

In this example, as the profit and loss are opposite in their nature, they must be distinguished by contrary signs. (Art. 39.) If the profit is marked $+$, the loss must be $-$.

Let $x=$ the sum required.

Then according to the statement, $x+350-60=200$

And $x=-90$.

Prob. 7. A ship sails 4 degrees north, then 13 S. then 17 N. then 19 S. and has finally 11 degrees of south latitude. What was her latitude at starting?

Prob. 8. If a certain number is divided by 12, the quotient, dividend, and divisor, added together, will amount to 64. What is the number?

Prob. 9. An estate is divided among four children, in such a manner that

The first has 200 dollars more than $\frac{1}{4}$ of the whole,
The second has 340 dollars more than $\frac{1}{5}$ of the whole,
The third has 300 dollars more than $\frac{1}{6}$ of the whole,
The fourth has 400 dollars more than $\frac{1}{8}$ of the whole,
What is the value of the estate?

Prob. 10. What is that number which is as much less than 500, as a fifth part of it is greater than 40?

Prob. 11. There are two numbers whose difference is 40, and which are to each other as 6 to 5. What are the numbers?

Prob. 12. Three persons, A, B and C draw prizes in a lottery. A draws 200 dollars; B draws as much as A, together with a third of what C draws; and C draws as much as A and B both. What is the amount of the three prizes?

Prob. 13. What number is that, which is to 12 increased by three times the number, as 2 to 9?

Prob. 14. A ship and a boat are descending a river at the same time. The ship passes a certain fort, when the boat is 13 miles below. The ship descends five miles, while the boat descends three. At what distance below the fort will they be together?

Prob. 15. What number is that, a sixth part of which exceeds an eighth part of it by 20?

Prob. 16. Divide a prize of 2000 dollars into two such parts, that one of them shall be to the other, as 9 : 7.

Prob. 17. What sum of money is that, whose third part, fourth part, and fifth part, added together, amount to 94 dollars?

Prob. 18. Two travellers, A and B, 360 miles apart, travel towards each other till they meet. A's progress is 10 miles an hour, and B's 8. How far does each travel before they meet?

Prob. 19. A man spent one-third of his life in England, one-fourth of it in Scotland, and the remainder of it, which was 20 years, in the United States. To what age did he live?

Prob. 20. What number is that, $\frac{1}{4}$ of which is greater than $\frac{1}{5}$ of it by 96?

Prob. 21. A post is $\frac{1}{5}$ in the earth, $\frac{3}{7}$ in the water, and 13 feet above the water. What is the length of the post?

Prob. 22. What number is that, to which 10 being added, $\frac{3}{5}$ of the sum will be 66?

Prob. 23. Of the trees in an orchard, $\frac{3}{4}$ are apple trees, $\frac{1}{10}$ pear trees, and the remainder peach trees, which are 20 more than $\frac{1}{8}$ of the whole. What is the whole number of trees in the orchard?

Prob. 24. A gentleman bought several gallons of wine for 94 dollars; and after using 7 gallons himself, sold $\frac{1}{4}$ of the remainder for 20 dollars. How many gallons had he at first?

Prob. 25. A and B have the same income. A contracts an annual debt amounting to $\frac{1}{7}$ of it; B lives upon $\frac{4}{5}$ of it; at the end of ten years, B lends to A enough to pay off his debts, and has 160 dollars to spare. What is the income of each?

Prob. 26. A gentleman lived single $\frac{1}{4}$ of his whole life; and after having been married 5 years more than $\frac{1}{7}$ of his life, he had a son who died 4 years before him, and who reached only half the age of his father. To what age did the father live?

Prob. 27. What number is that, of which if $\frac{1}{3}$, $\frac{1}{4}$ and $\frac{2}{7}$ be added together the sum will be 73?

Prob. 28. A person after spending 100 dollars more than $\frac{1}{3}$ of his income, had remaining 35 dollars more than $\frac{1}{2}$ of it. Required his income.

Prob. 29. In the composition of a quantity of gunpowder,
The *nitre* was 10 lbs. more than $\frac{2}{3}$ of the whole.
The *sulphur* $4\frac{1}{2}$ lbs. less than $\frac{1}{6}$ of the whole.
The *charcoal* 2 lbs. less than $\frac{1}{7}$ of the nitre.
What was the amount of gunpowder?

Prob. 30. A cask which held 146 gallons, was filled with a mixture of brandy, wine and water. There were 15 gallons of wine more than of brandy, and as much water as the brandy and wine together. What quantity was there of each?

Prob. 31. Four persons purchased a farm in company for 4755 dollars; of which B paid three times as much as A; C paid as much as A and B; and D paid as much as C and B. What did each pay?

Prob. 32. It is required to divide the number 99 into five such parts, that the first may exceed the second by 3, be less than the third by 10, greater than the fourth by 9, and less than the fifth by 16.

Prob. 33. A father divided a small sum among four sons.
The third had 9 shillings more than the fourth;
The second had 12 shillings more than the third;
The first had 18 shillings more than the second;
And the whole sum was 6 shillings more than 7 times the sum which the youngest received.
What was the sum divided?

Prob. 34. A farmer had two flocks of sheep, each containing the same number. Having sold from one of these 39, and from the other 93, he finds twice as many remaining in the one as in the other. How many did each flock originally contain?

Prob. 35. An express, travelling at the rate of 60 miles a day, had been despatched 5 days, when a second was sent after him, travelling 75 miles a day. In what time will the one overtake the other?

Prob. 36. The age of A is double that of B, the age of B triple that of C, and the sum of all their ages 140. What is the age of each?

Prob. 37. Two pieces of cloth, of the same price by the yard, but of different lengths, were bought, the one for £5, the other for £6$\frac{1}{2}$. If 10 be added to the length of each, the sums will be as 5 to 6. Required the length of each piece.

Prob. 38. A and B began trade with equal sums of money. The first year, A gained forty pounds, and B lost 40. The second year, A lost $\frac{1}{3}$ of what he had at the end of the first, and B gained 40 pounds less than twice the sum which A had lost. B had then twice as much money as A. What sum did each begin with?

Prob. 39. What number is that, which being severally added to 36 and 52, will make the former sum to the latter, as 3 to 4?

Prob. 40. A gentleman bought a chaise, horse and harness for 360 dollars. The horse cost twice as much as the harness; and the chaise cost twice as much as the harness and horse together. What was the price of each?

Prob. 41. Out of a cask of wine, from which had leaked $\frac{1}{3}$ part, 21 gallons were afterwards drawn; when the cask was found to be half full. How much did it hold?

Prob. 42. A man has 6 sons, each of whom is 4 years older than his next younger brother; and the eldest is three times as old as the youngest. What is the age of each?

Prob. 43. Divide the number 49 into two such parts, that the greater increased by 6, shall be to the less diminished by 11, as 9 to 2.

Prob. 44. What two numbers are as 2 to 3; to each of which, if 4 be added, the sums will be as 5 to 7?

Prob. 45. A person bought two casks of porter, one of which held just 3 times as much as the other; from each of these he drew 4 gallons, and then found that there were 4 times as many gallons remaining in the larger, as in the other. How many gallons were there in each?

Prob. 46. Divide the number 68 into two such parts, that the difference between the greater and 84, shall be equal to 3 times the difference between the less and 40.

Prob. 47. Four places are situated in the order of the letters A, B, C, D. The distance from A to D is 34 miles. The distance from A to B is to the distance from C to D as 2 to 3. And $\frac{1}{4}$ of the distance from A to B, added to half the distance from C to D, is three times the distance from B to C. What are the respective distances?

Prob. 48. Divide the number 36 into 3 such parts, that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third, shall be equal to each other.

Prob. 49. A merchant supported himself 3 years, for £50 a year, and at the end of each year added to that part of his stock which was not thus expended, a sum equal to one-third of this part. At the end of the third year his original stock was doubled. What was that stock?

Prob. 50. A general having lost a battle, found that he had only half of his army $+3600$ men left fit for action; $\frac{1}{8}$ of the army $+600$ men being wounded; and the rest, who were $\frac{1}{5}$ of the whole, either slain, taken prisoners, or missing. Of how many men did his army consist?

SECTION VIII.

INVOLUTION.

Art. 163. Definitions.—(1.) *When a quantity is multiplied into itself, the product is called a power.* Thus $3\times3=9$; and $d\times d=dd$. The 9 and dd are powers of 3 and d.

(2.) *Powers are divided into different orders or degrees, as the first, second, third, fourth, fifth powers, &c., which are also called the square, cube, biquadrate, &c.*

Quest.—What is a power? How are powers divided? What is the second power called? Third? Fourth?

They take their name from the number of times the *root*, or *first power*, is used as a *factor* in producing the given power.

The original quantity is called the *first power* or *root* of all the other powers, because they are all derived from it.

Thus, $2\times2=4$, the square or second power of 2.

$2\times2\times2=8$, the cube or third power.

$2\times2\times2\times2=16$, the biquadrate or fourth power, &c.

And $a\times a=aa$, the second power of a.

$a\times a\times a=aaa$, the third power.

$a\times a\times a\times a=aaaa$, the fourth power, &c.

(3.) The *number* of times a quantity is employed as a *factor* to produce the given power, is generally indicated by a *figure* or *letter* placed above it on the right hand. This figure or letter is called the *index* or *exponent*. Thus $a\times a$ is written a^2 instead of aa; and $a\times a\times a=a^3$.

The *index* of the *first power* is 1; *but this is commonly omitted*, for $a^1=a$.

Obser. An index is totally different from a *co-efficient*. The *latter* shows how many times a quantity is taken as a *part* of a whole; the *former* how many times the quantity is taken as a *factor*. Thus $4a=a+a+a+a$; but $a^4=a\times a\times a\times a=aaaa$. If $a=4$, then $4a=16$; and $a^4=256$.

(4.) Powers are also divided into *direct* and *reciprocal*.

Direct Powers are those which have *positive* indices, as d^2, d^5, &c., and are produced by multiplying a quantity into itself Thus $d\times d=d^2$; $d\times d\times d=d^3$; and $d\times d\times d\times d=d^4$.

A *Reciprocal Power* of a quantity is the quotient arising from dividing a *unit* by the *direct* power of that quantity, as $\frac{1}{d^2}$, $\frac{1}{d^3}$, $\frac{1}{d^4}$, &c. (Art. 32.)

QUEST.—From what do they take their name? What is the first power? How are powers denoted? What is this number called? What does it show? What is the difference between an index and a co-efficient? What is the index of the first power? Is it usually written? How else are powers divided? What are direct powers? Reciprocal powers?

It is produced by dividing a *direct power* by its *root*, till we come to the root itself; and then *continuing* the division, we obtain the *reciprocal* powers. Thus $\frac{d^3}{d}=d^2$; and $\frac{d^2}{d}=d$; and $\frac{d}{d}=1$; and $\frac{1}{d}\div d=\frac{1}{d^2}$; and $\frac{1}{d^2}\div d=\frac{1}{d^3}$, &c.

For convenience of calculation, reciprocal powers are written like direct powers with the sign $-$ before the *index;* thus $\frac{1}{d^2}=d^{-2}$, &c. The direct and reciprocal powers of d, are d^4, d^3, d^2, d^1, d^0, d^{-1}, d^{-2}, d^{-3}, d^{-4}, &c.

164. INVOLUTION *is the process* of finding any power of a quantity by multiplying it into itself. Hence,

165. To involve a quantity to any required power.

RULE.—*Multiply the quantity into itself, till it is taken as a factor as many times as there are units in the index of the power to which the quantity is to be raised.* (Art. 80.)

N. B. All powers of 1 are the same, viz. 1. For $1\times1\times1\times1$, &c. $=1$.

166. A *single letter* is involved, by giving it the *index* of the proposed power; or by repeating it as a factor as many times as there are units in that index.

N. B. If the letter or quantity has a *co-efficient*, it must be raised to the *required power* by *actual multiplication.*

1. The 4th power of a, is a^4, or $aaaa$. (Arts. 163, 165.)
2. The 6th power of y, is y^6, or $yyyyyy$.
3. The nth power of x, is x^n, or $xxx\ldots n$ times repeated.
4. Required the 3d power of $3x$.

QUEST.—How are reciprocal powers written? What is involution? The rule? What are all powers of 1? How is a single letter involved? If the quantity has a co-efficient, what must be done with it?

5. Required the 4th power of $4y$.

6. Required the 7th power of $2a$.

167. The method of involving a quantity which consists of several *factors*, depends on the principle, that *the power of the product of several factors is equal to the product of their powers.*

7. Thus $(ay)^2=a^2y^2$. For by Art. 164, $(ay)^2=ay\times ay$.
But $ay\times ay=ayay=aayy=a^2y^2$.

8. What is the 3d power of bmx?

9. What is the nth power of ady?

In finding the power of a *product*, therefore, we may either involve the *whole* at *once;* or we may involve *each* of the factors *separately*, and then multiply their several powers into each other.

10. What is the 4th power of dhy?

11. What is the 3d power of $4b$?

12. What is the nth power of $6ad$?

13. What is the 3d power of $3m\times 2y$?

168. SIGNS.—*When the root is positive, all its powers are positive also; but when the root is negative, the* ODD *powers are negative, while the* EVEN *powers are positive.* (Art. 82.)

169. Hence any *odd* power has the same sign as its root. But an *even* power is positive, whether its root is positive or negative. Thus $+a\times+a=a^2$. And $-a\times-a=a^2$.

170. To involve a quantity which is already a power.

Multiply the index of the quantity into the index of the power to which it is to be raised.

QUEST.—On what principle does the method of involving a quantity which consists of several factors, depend? How then may we find the power of a product? Rule for signs? Does this differ from the rule for signs in multiplication? What sign has every odd power? Even powers? How involve a quantity which is already a power?

14. The 3d power of a^2, is $a^{2 \cdot 3} = a^6$.

For $a^2 = aa$: and the cube of aa is $aa \times aa \times aa = aaaaaa = a^6$; which is the 6th power of a, but the third power of a^2.

15. Find the 4th power of a^3b^2.

16. Find the 3d power of $4a^2x$.

17. Find the 4th power of $2a^3 \times 3x^2d$.

18. Find the 5th power of $(a+b).^2$

19. Find the 2d power of $(a+b)^n$.

20. Find the nth power of $(x-y)^m$.

21. Find the nth power of $(x+y)^2$.

22. Find the 2d power of $(a^3 \times b^3)$.

23. Find the 3d power of $(a^3b^2h^4)$.

171. *A* FRACTION *is raised to a power by involving both the numerator and the denominator.*

24. The square of $\frac{a}{b}$ is $\frac{a^2}{b^2}$. For, by the rule for the multiplication of fractions, $\frac{a}{b} \times \frac{a}{b} = \frac{aa}{bb} = \frac{a^2}{b^2}$. (Art. 130.)

25. Find the 2d, 3d and nth powers of $\frac{1}{a}$.

26. Find the cube of $\frac{2xr^2}{3y}$.

27. Find the nth power of $\frac{x^2r}{ay^m}$.

28. Find the square of $\frac{-a^3 \times (d+m)}{(x+1)^3}$.

172. A compound quantity consisting of terms connected by + and —, is involved by an actual multiplication of its several parts. Thus,

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QUEST.—How is a fraction involved? How is a compound quantity involved?

29. $(a+b)^1=a+b$, ... the first power.

$$\begin{array}{l}
\quad a+b \\
\hline
a^2+ab \\
\quad\;\; +ab+b^2 \\
\hline
\end{array}$$

$(a+b)^2=a^2+2ab+b^2$, ... the second power.

$$\begin{array}{l}
a\;+b \\
\hline
a^3+2a^2b+\;ab^2 \\
\quad\;\; +\;a^2b+2ab^2+b^3 \\
\hline
\end{array}$$

$(a+b)^3=a^3+3a^2b+3ab^2+b^3$, . . the third power.

$$\begin{array}{l}
a\;+\;b \\
\hline
a^4+3a^3b+3a^2b^2+\;ab^3 \\
\quad\;\; +\;a^3b+3a^2b^2+3ab^3+b^4 \\
\hline
\end{array}$$

$(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4$, fourth power.

30. Find the square of $a-b$.

31. Find the cube of $a+1$.

32. Find the square of $a+b+h$.

33. Required the cube of $a+2d+3$.

34. Required the 4th power of $b+2$.

35. Required the 5th power of $x+1$.

36. Required the 6th power of $1-b$.

173. The squares of *binomial* and *residual* quantities occur so frequently in algebraic processes, that it is important to make them familiar.

If we multiply $a+h$ into itself, and also $a-h$ into itself,

37. We have $a+h$

$$\begin{array}{l}
\quad a+h \\
\hline
a^2+ah \\
\quad\;\; +ah+h^2 \\
\hline
a^2+2ah+h^2.
\end{array}$$

38: And $a-h$

$$\begin{array}{l}
\quad a-h \\
\hline
a^2-ah \\
\quad\;\; -ah+h^2 \\
\hline
a^2-2ah+h^2.
\end{array}$$

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Here it will be seen, that in each case, the first and last terms are squares of a and h; and that the middle term is twice the product of a into h. Hence the squares of binomial and residual quantities, without multiplying each of the terms separately, may be found, by the following proposition.*

(1.) *The square of a* BINOMIAL, *the terms of which are both positive, is equal to the square of the first term,* + *twice the product of the two terms,* + *the square of the last term.*

(2.) *The square of a* RESIDUAL *quantity, is equal to the square of the first term,* — *twice the product of the two terms,* + *the square of the last term.*

39. Find the square of $2a+b$.

40. Find the square of $h+1$.

41. Find the square of $ab+cd$.

42. Find the square of $6y+3$.

43. Find the square of $3d-h$.

44. Find the square of $a-1$.

174. For many purposes it will be sufficient to express the powers of compound quantities by *exponents* without an actual multiplication.

45. Thus the square of $a+b$, is $\overline{a+b}|^2$, or $(a+b)^2$.

46. Find the nth power of $bc+8+x$.

In cases of this kind, the vinculum must be drawn over *all* the terms of which the compound quantity consists.

175. But if the root consists of several *factors*, the vinculum which is used in expressing the power, may either extend over the whole; or may be applied to each of the factors separately, as convenience may require.

QUEST.—What is the square of a binomial whose signs are plus? Of a residual? Is it always necessary to perform the multiplication? How far must the vinculum extend when the root contains factors?

* Euclid, 2. 4.

47. Thus the square of $(a+b)\times(c+d)$, is either

$$\overline{(a+b)\times(c+d)}^2, \text{ or } (a+b)^2\times(c+d)^2.$$

For the first of these expressions is the square of the product of the two factors, and the last is the product of their squares. But one of these is equal to the other. (Art. 167.)

The cube of $a\times(b+d)$, is $\overline{a\times(b+d)}^3$, or $a^3\times(b+d)^3$.

176. When a quantity whose power has been expressed by a vinculum and an index, is afterwards involved by an actual multiplication of the terms, it is said to be *expanded.*

48. Thus $(a+b)^2$, when expanded, becomes $a^2+2ab+b^2$.

49. Expand $(a+b+h)^2$.

BINOMIAL THEOREM.*

177. To involve a *binomial* to a *high power* by actual multiplication, as in Art. 172, is a *long* and *tedious process.* A much *easier* and *more expeditious* way to obtain the required power, is by what is called the BINOMIAL THEOREM. This ingenious and beautiful method was invented by SIR ISAAC NEWTON, and has been deemed of so *great importance* to mathematical investigation, that *it is inscribed on his monument* in Westminster Abbey.

178. To illustrate *this theorem*, let the pupil involve the binomial $a+b$, (Art. 172,) and the residual $a-b$, to the 2d, 3d and 4th powers. Thus,

$(a-b)^2=a^2-2ab+b^2$.

$(a-b)^3=a^3-3a^2b+3ab^2-b^3$.

$(a-b)^4=a^4-4a^3b+6a^2b^2-4ab^3+b^4$.

QUEST.—What is meant by expanding a quantity? What is the best mode of involving a binomial to a high power? Who is the author of this theorem? In what light is it regarded? What is $(a+b)^2$? $(a+b)^3$? $(a+b)^4$? $(a-b)^2$? $(a-b)^3$? $(a-b^4)$?

* See Preface.

179. By a careful inspection of the several parts of the preceding work, the following particulars will be observed to be *common* to *each power*.

I. By counting the terms it will be found that the *number* in each power is *greater* by 1 than the *index* of that power; e. g. in the 3d power the number of terms is 4; in the 4th power, it is 5, &c.

II. If we examine the signs we shall perceive when both terms of the binomial are *positive*, that all the signs in every power are +; but when the quantity is a *residual*, all the *odd* terms, reckoning from the left, have the sign +, and all the *even* terms have the sign —. Thus in the 4th power, the signs of the *first*, *third* and *fifth* terms are +, while those of the *second* and *fourth* are —.

III. Let us now direct our attention to the *indices*.

1. It will be seen that the index of the *first* term, or the *leading quantity** in each power, always begins with the index of the *proposed power*, and *decreases* 1 in each successive term towards the right, till we come to the *last* term from which the letter itself is excluded. Thus in $(a \pm b)^4$ the indices of the leading quantity a, are 4, 3, 2, 1.

2. The index of the *following* quantity begins with 1 in the second term, and *increases* regularly by 1 to the *last* term, whose index, like that of the first, is the index of the *required power*. Thus in $(a \pm b)^4$ the indices of the following quantity b, are 1, 2, 3, 4.

3. We shall also perceive, that the *sum* of the indices is the *same* in *each* term of any given power; and this sum is

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QUEST.—How many terms are there in each power? What signs has a binomial? Residual? What are the indices of the leading quantity? Of the following quantity? Which is the leading quantity? The following? To what is the sum of the indices in each term equal?

* The *first* letter of a binomial is called the *leading* quantity, and the *other*, the *following* quantity.

equal to the *index* of *that* power. Thus the sum of the indices in each of the terms of the 4th power, is 4.

IV. The last thing to be considered is the *co-efficients* of the several terms.

1. The *co-efficient* of the *first* and *last* terms in each power, is 1; the co-efficient of the *second* and *next* to the last terms, is the *index* of the required power. Thus in the 3d power, the index of the second and next to the last terms, is 3; and in the same terms in the 4th power, it is 4, &c.

2. It will be observed also, that the co-efficients *increase* in a regular manner through the *first half* of the terms; and then *decrease* at the *same* rate through the last half. Thus,

in the 4th power they are 1, 4, 6, 4, 1,
in the 6th power they are 1, 6, 15, 20, 15, 6, 1.

3. The co-efficients of *any* two terms *equally distant* from the extremes, are *equal* to each other. Thus in the 4th power, the second term from each extreme is 4; in the 6th power, the second term from each extreme is 6, and the third is 15.

4. The sum of all the co-efficients in each power, is equal to the number 2 raised to that power. Thus $(2)^4=16$; also, the sum of the co-efficients in the 4th power, is 16, and $(2)^6=64$; so the sum of the co-efficients in the 6th power, is 64.

180. If we involve *any other binomial, or residual, to any required power whatever*, we shall find the foregoing principles are *true in all cases*, and will apply to

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QUEST.—What is the co-efficient of the first and last term? What of the second and next to the last? What is peculiar to the first half of them? To the last half? How do those equally distant from the extremes compare? To what is the sum of all the co-efficients in any power equal? What is said as to the extent of the foregoing principles? What then do they furnish?

all examples. Hence we may safely conclude, that they are *universal principles*, and may be employed in raising all binomials to any required power. They are the *basis*, or *elements* of what is called the *Binomial Theorem*.

181. The BINOMIAL THEOREM may be defined, *a general method of involving binomial quantities to any proposed power.* It is comprised in the following

GENERAL RULE.

I. SIGNS.—*If both terms of the binomial have the sign* $+$, *all the signs in every power will be* $+$; *but if the given quantity is a residual, all the odd terms in each power, reckoning from the left, will have the sign* $+$, *and the even terms the sign* $-$.

II. INDICES.—*The* INDEX *of the first term or leading quantity, must always be the index of the required power; and this decreases regularly by* 1 *through the other terms. The index of the following quantity begins with* 1 *in the second term, and increases regularly by* 1 *through the others.*

III. CO-EFFICIENTS.—*The co-efficient of the first term is* 1; *that of the second is equal to the index of the power; and universally, if the co-efficient of any term be multiplied by the index of the leading quantity in that term, and divided by the index of the following quantity increased by* 1, *it will give the co-efficient of the succeeding term.*

IV. *The number of terms will always be one greater than the power required.*

In algebraic characters, the theorem is this,

$$(a+b)^n=a^n+a\times a^{n-1}b+n\times\frac{n-1}{2}a^{n-2}b^2, \text{ \&c.}$$

QUEST.—What is the Binomial Theorem? What is the rule for the signs? For the indices? For the co-efficients? The number of terms?

N. B. It is here supposed, that the *terms* of the binomial have no other co-efficients or exponents than 1. Other binomials may be reduced to this form by substitution. (Art. 159.)

1. What is the 6th power of $x+y$?

The terms without the co-efficients, are

$$x^6,\ x^5y,\ x^4y^2,\ x^3y^3,\ x^2y^4,\ xy^5,\ y^6.$$

And the coefficients, are

$$1,\quad 6,\quad \frac{6\times5}{2},\quad \frac{15\times4}{3},\quad \frac{20\times3}{4},\quad 6,\quad 1.$$

that is $\quad 1,\quad 6,\quad 15,\quad 20,\quad 15,\quad 6,\quad 1.$

Prefixing these to the several terms, we have the power required;

$x^6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6$. Ans.

2. What is the 5th power of $(d+h)$?

3. What is the nth power of $(b+y)$?

Ans. $b^n+Ab^{n-1}y+Bb^{n-2}y^2+Cb^{n-3}y^3+Db^{n-4}y^4$, &c.

That is, supplying the co-efficients which are here represented by A, B, C, &c.

$$b^n+n\times b^{n-1}y+n\times\frac{n-1}{2}\times b^{n-2}y^2,\ \&c.$$

4. What is the 5th power of x^2+3y^2?

Substituting a for x^2, and b for $3y^2$, (Art. 159,) we have

$(a+b)^5=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$.

And restoring the values of a and b,

$(x^2+3y^2)^5=x^{10}+15x^8y^2+90x^6y^4+270x^4y^6+405x^2y^8+243y^{10}$.

5. What is the 6th power of $(3x+2y)$?

6. What is the 2d power of $(a-b)$?

7. What is the 3d power of $(a-b)$?

QUEST.—Can this rule be applied to binomials whose co-efficients exceed 1? How?

8. What is the 4th power of $(a-b)$?

9. What is the 6th power of $(x-y)$?

10. What is the nth power of $(a-b)$?

182. When one of the terms of a binomial is a *unit*, it is generally omitted in the power, except in the first or last term; because every power of 1 is 1, (Art. 165,) and this when it is a factor, has no effect upon the quantity with which it is connected. (Art. 70.)

11. Thus the cube of $(x+1)$ is $x^3+3x^2\times 1+3x\times 1^2+1^3$,
Which is the same as x^3+3x^2+3x+1.

12. What is the 4th power of $(a-1)$?

The insertion of the powers of 1 is of no use, unless it be to preserve the *exponents* of both the leading and the following quantity in each term, for the purpose of finding the co-efficients. But this will be unnecessary, if we bear in mind, that the *sum* of the two exponents, in each term, is equal to the index of the power. (Art. 179, 3.) So that, if we have the exponent of the *leading* quantity, we may know that of the *following* quantity, and v. v.

13. What is the 6th power of $(1-y)$?

14. What is the nth power of $(1+x)$?

183. The binomial theorem may also be applied to quantities consisting of *more than two terms*. By substitution, several terms may be reduced to two, and when the compound expressions are restored, such of them as have exponents may be separately expanded. (Art. 159.)

15. What is the cube of $a+b+c$?

Substituting h for $(b+c)$, we have $a+(b+c)=a+h$.

And by the theorem, $(a+h)^3=a^3+3a^2h+3ah^2+h^3$.

QUEST.—When one of the terms of a binomial is a *unit*, how proceed? Can a binomial theorem be applied to quantities which have more than two terms? How?

That is, restoring the value of h,

$$(a+b+c)^3 = a^3 + 3a^2 \times (b+c) + 3a \times (b+c)^2 + (b+c)^3.$$

The last two terms contain powers of $(b+c)$; but these may be separately involved.

183.*a*. Binomials, in which one of the terms is a fraction, may be involved by actual multiplication; or by reducing the given quantity to an improper fraction, and then involving the fraction according to Art. 171.

16. Find the square of $x+\frac{1}{2}$; and $x-\frac{1}{2}$, as in Art. 173.

$$\begin{array}{ll} x+\frac{1}{2} & \\ x+\frac{1}{2} & \\ \hline x^2+\frac{1}{2}x & \\ \quad\ +\frac{1}{2}x+\frac{1}{4} & \\ \hline x^2+x+\frac{1}{4} & \end{array} \qquad\qquad \begin{array}{ll} x-\frac{1}{2} & \\ x-\frac{1}{2} & \\ \hline x^2-\frac{1}{2}x & \\ \quad\ -\frac{1}{2}x+\frac{1}{4} & \\ \hline x^2-x+\frac{1}{4}. & \end{array}$$

Or, reduce the mixed quantities to improper fractions.

Thus, $\quad x+\frac{1}{2}=\frac{2x+1}{2}$; and $x-\frac{1}{2}=\frac{2x-1}{2}$. (Art. 120.)

$$\left(\frac{2x+1}{2}\right)^2=\frac{4x^2+4x+1}{4}; \text{ and } \left(\frac{2x-1}{2}\right)=\frac{4x^2-4x+1}{4}.$$

17. Find the square of $a+\dfrac{2}{3}$.

18. Find the square of $x-\dfrac{b}{2}$.

19. Find the square of $-\dfrac{b}{m}+3xy$.

20. Find the square of $-\dfrac{6}{7}+2abc$.

QUEST.—How involve a binomial when one term is a fraction?

EXAMPLES FOR PRACTICE.

1. Expand $(x+y)^3$.
2. Expand $(a+b)^4$.
3. Expand $(a-b)^6$.
4. Expand $(x+y)^5$.
5. Expand $(x-y)^8$.
6. Expand $(m+n)^7$.
7. Expand $(a+b)^9$.
8. Expand $(x+y)^{10}$.
9. Expand $(x-y)^{13}$.
10. Expand $(a-b)^7$.
11. Expand $(a+b)^8$.
12. Expand $(2+x)^5$.
13. Expand $(a-bx+c)^3$.
14. Expand $(a+3bc)^3$.
15. Expand $(2ab-x)^4$.
16. Expand $(4ab-5c)^2$.
17. Expand $(3x-6y)^3$.
18. Expand $(5a+3d)^3$.

ADDITION OF POWERS.

184. It is obvious that powers may be added, like other quantities, *by writing them one after another, with their signs.* (Art. 47.)

1. Thus the sum of a^3 and b^2, is a^3+b^2.

2. And the sum of a^2-b^n and h^5-d^4, is $a^2-b^n+h^5-d^4$.

185. *The same powers of the same letters* are *like quantities,* (Art. 28;) hence their co-efficients may be added or subtracted, as in Arts. 50 and 51.

3. Thus the sum of $2a^2$ and $3a^2$, is $5a^2$.

	4	5.	6.	7.	8
To	$-3x^6y^5$	$3b^m$	$3a^4y^n$	$-5a^3h^6$	$3(a+y)^n$
Add	$-2x^6y^5$	$6b^m$	$-7a^4y^n$	$6a^3h^6$	$5(a+y)^n$

185.a. But powers of *different letters,* and *different* powers of the *same* letter, are *unlike quantities,* (Art. 28;) hence they

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QUEST.—What is the general method of adding powers? How are the *same powers* of the *same letters* added? How are powers of *different letters,* and *different powers* of the *same letter,* added?

can be added only by writing them down with their signs. (Art. 55.)

9. The sum of a^2 and a^3, is a^2+a^3.

It is evident that the square of a, and the cube of a, are neither twice the square of a, nor twice the cube of a.

10. The sum of a^3b^n and $3a^5b^6$, is $a^3b^n+3a^5b^6$.

186. From the preceding principles we deduce the following

GENERAL RULE FOR ADDING POWERS.

If the powers are like quantities, add their co-efficients, and to the sum annex the common letter or letters with their given indices.

II. *If the powers are unlike quantities, they must be added by writing them, one after another, without altering their signs.*

11. Add $5x(a-b)^3+x(a-b)^3$ to $2x(a-b)^3+10x(a-b)^3$.
12. Add $3(x+y)^4+5a^2-4(x+y)^4$ to $10a^2+6(x+y)^4$.
13. Add $a^3b^2+x^6y^4+a^2b^3$ and $-x^2y^4+a^4b^6$.
14. Add $5a^2bc^3$, $3a^2bc^3$, a^2bc^3 and $2a^2bc^3$.
15. Add $3a^3+bc^2+5a^3+2bc^2$ and a^3+5bc^2 to $6a^3+2bc^2$.
16. Add $\frac{1}{2}(xy-cm)^6$, $3(xy-cm)^6$, $-\frac{1}{4}(xy-cm)^6$ and $\frac{3}{4}(xy-cm)^6$.

SUBTRACTION OF POWERS.

187. Rule.—*Subtraction of powers is performed in the same manner as addition, except that the signs of the subtrahend must be changed as in simple subtraction.* (Art. 60.)

Quest.—General rule for adding powers? How are powers subtracted?

1. From $2a^4$ take $-6a^4$. Ans. $8a^4$.

2. From	$-3b^n$	3. $3h^2b^6$	4. a^3b^n	5. $5(a-h)_6$
Take	$4b^n$	$4h^2b^6$	a^3b^n	$2(a-h)_6$

6. From $6a(a+b)^4$ take $a(a+b)^4$.

7. From $17a^2x^3+5xy^2$ take $12a^2x^3-4xy^2$.

8. From $3a^3(b^2-8)^3$ take $a^3(b^2-8)^3$.

9. From $a^2b^3+x^3y^4$ take $a^5b^6-x^2y^3$.

10. From $5(x^3+y^4)^3-3(a^2-b^3)^5$ take $-3(a^2-b^3)^3+4(x^3+y^4)^3$.

11. From $2x(a-b)^3+3(a-b)^3$ take $x(a-b)^3+3(a-b)^3$.

12. From $\frac{1}{2}(x+y)^2+\frac{1}{3}(a+b)^3$ take $\frac{1}{4}(x+y)^2+\frac{2}{6}(a+b)^3$.

MULTIPLICATION OF POWERS.

188. *Powers* may be multiplied, like other quantities, by writing the factors one after another, either with, or without, the sign of multiplication between them. (Art. 72.)

1. The product of a^3 into b^2 is a^3b^2; and x^{-3} into a^m is a^mx^{-3}.

2. Mult.	h^2b^{-n}	3. $3a^6y^2$	4. dh^3x^{-n}	5. $a^2b^3y^2$
Into	a^4	$-2x$	$4by^4$	a^3b^2y

188.*a*. If the quantities to be multiplied are *powers of the same root*, instead of writing the factors one after another, as in the last article, we may *add their exponents*, and the *sum* placed at the right hand of the root will be the *product* required.

The reason of this operation may be illustrated thus:

$a^2\times a^3$ is a^2a^3, (Art. 188.) But $a^2=aa$, and $a^3=aaa$.

And $aa\times aaa=aaaaa=a^5$, (Art. 80.) The sum of the exponents $2+3$, is also 5. So $d^m\times d^n=d^{m+n}$.

N. B. The same principles hold true in all other powers of the same root.

189. Hence we deduce the following

GENERAL RULE FOR MULTIPLYING POWERS.

I. *Powers of the same root may be multiplied by adding their exponents.* (Art. 188.a.)

II. *If the powers have co-efficients, these must be multiplied together, and their product prefixed to the common letter or letters.*

III. *Powers of different roots are multiplied by writing them one after another, either with, or without, the sign of multiplication between them.* (Art. 188.)

Thus $a^2 \times a^6 = a^{2+6} = a^8$. And $x^3 \times x^2 \times x = x^{3+2+1} = x^6$.

	6.	7.	8.	9.	10.
Mult.	$4a^n$	$3x^4$	b^2y^3	$a^2b^3y^2$	$(b+h-y)^n$
Into	$2a^n$	$2x^3$	b^4y	a^3b^2y	$(b+h-y)$

11. Mult. $x^3+x^2y+xy^2+y^3$ into $x-y$.

12. Mult. $4x^2y+3xy-1$ into $2x^2+x$.

13. Mult. x^3+x-5 into $2x^2+x+1$.

190. The rule is equally applicable to powers whose exponents are *negative*, i. e. to reciprocal powers.

14. Thus $a^{-2} \times a^{-3} = a^{-5}$. That is, $\frac{1}{aa} \times \frac{1}{aaa} = \frac{1}{aaaaa}$.

15. Mult. y^n into y^m into y^4.

16. Mult. a^{-2} into a^{-3} into a^{-8}.

17. Mult. a^{-2} into a^3 into $-a^{-5}$.

18. Mult. a^{-n} into a^m into $-a^{2n}$.

19. Mult. y^{-2} into y^2 into $-y^ny^3$.

QUEST.—How are powers of the same root multiplied? Of different roots? When the powers have co-efficients, what must be done with them? Is this rule applicable to reciprocal powers?

20. If $a+b$ be multiplied into $a-b$, the product will be a^2-b^2, (Art. 86;) that is,

191. *The product of the sum and difference of two quantities, is equal to the difference of their squares.*

This is an instance of the facility with which *general truths* are demonstrated in algebra.

If the sum and difference of the *squares* be multiplied, the product will be equal to the difference of the *fourth* powers, &c.

21. Mult. $(a-y)$ into $(a+y)$.
22. Mult. (a^2-y^2) into (a^2+y^2).
23. Mult. (a^4-y^4) into (a^4+y^4).
24. Mult. $a^2+a^4+a^6$ into a^2-1.
25. Mult. $3a(x^2-y^3)^3$ into $2a(x^2-y^3)^4$.
26. Mult. $\frac{1}{2}(a^2+b^3)^3$ into $\frac{1}{2}(a^2+b^3)^2$.
27. Mult. a^3-b^2 into a^3+b^2.
28. Mult. $x^3+x^2y+xy^2+y^3$ into $x+y$.
29. Mult. $a^4-2a^3b+4a^2b^2-8ab^3-16b^4$ into $a+2b$
30. Mult. a^2+b into a^2-8.

DIVISION OF POWERS.

192. Powers may be divided, like other quantities, by rejecting from the dividend a factor equal to the divisor; or by placing the divisor under the dividend, in the form of a fraction. Thus the quotient of a^3b^2 divided by b^2, is a^3.

	1.	2.	3	4.
Div.	$9a^3y^4$	$12b^3x^n$	$a^2b+3a^2y^4$	$d\times(a-h+y)^3$
By	$-3a^3$	$2b^3$	a^2	$(a-h+y)^2$

QUEST.—What is the product of the sum and difference of two quantities equal to?

5. The quotient of a^5 divided by a^3, is $\frac{a^5}{a^3}$. But this is equal to a^2. For, in the series

$$a^{+4},\ a^{+3},\ a^{+2},\ a^{+1},\ a^0,\ a^{-1},\ a^{-2},\ a^{-3},\ a^{-4},\ \&c.,$$

if any term be divided by another, the index of the quotient will be equal to the *difference* between the index of the dividend and that of the divisor.

Thus $a^5 \div a^3 = \frac{aaaaa}{aaa} = a^2$. And $a^m \div a^n = \frac{a^m}{a^n} = a^{m-n}$.

193. Hence we deduce the following

GENERAL RULE FOR DIVIDING POWERS.

I. *A power may be divided by another power of the same root by subtracting the index of the divisor from that of the dividend.*

II. *If the divisor and dividend have co-efficients, the co-efficient of the dividend must be divided by that of the divisor.* (Art. 96.)

III. *If the divisor and dividend are both compound quantities, the terms must be arranged, and the operation conducted in the same manner, as in simple division of compound quantities.* (Art. 107.)

6. Thus $y^3 \div y^2 = y^{3-2} = y^1$. That is, $\frac{yyy}{yy} = y$.

7. Divide a^{n+1} by a. 8. Divide x^n by x^n.

	9.	10.	11.	12.	13.
Divide	y^{2m}	b^6	$8a^{n+m}$	a^{n+3}	$12(b+y)^n$
By	y^m	b^3	$4a^m$	a^2	$3(b+y)^3$

194. The rule is equally applicable to reciprocal powers.

QUEST.—How is a power divided by another power of the same root? If the divisor and dividend have co-efficients, how proceed? When they are both compound quantities, how? Is this rule applicable to reciprocal powers?

Thus the quotient of a^{-5} by a^{-3}, is a^{-2}.

That is $\frac{1}{aaaaa} \div \frac{1}{aaa} = \frac{1}{aaaaa} \times \frac{aaa}{1} = \frac{aaa}{aaaaa} = \frac{1}{aa}$.

15. Divide $-x^{-5}$ by x^{-3}.
16. Divide h^2 by h^{-1}.
17. Divide $6a^n$ by $2a^{-3}$
18. Divide ba^3 by a.
19. Divide b^3 by b^5.
20. Divide a^4 by a^7.
21. Divide $(a^3+y^3)^m$ by $(a^3+y^3)^n$.
22. Divide $(b+x)^n$ by $(b+x)$.

Examples of compound divisors with indices. (Art. 105.)

23. Divide a^3+x^3 by $a+x$.
24. Divide a^4+4x^4 by $a^2-2ax+2x^2$.
25. Divide x^6-1 by $x-1$.
26. Divide $a^4+2a^3b+2a^2b^2+ab^3$ by $a^3+a^2b+ab^2$.
27. Divide b^8-16c^8 by b^2-2c^2.
28. Divide $a^6-a^4x-a^2x^3+2x^4$ by a^4-x^3.
29. Divide $a^4+4a^3b+6a^2b^2+4ab^3+b^4$ by $a^2+2ab+b^2$.
30. Divide $8x^3-y^3$ by $2x-y$.
31. Divide $x^3-3ax^2+3a^2x-a^3$ by $x-a$.
32. Divide $2y^3-19y^2+26y-16$ by $y-8$.
33. Divide x^6-1 by $x+1$.
34. Divide $4x^4-9x^2+6x-3$ by $2x^2+3x-1$.
35. Divide $a^4+4a^2b+3b^4$ by $a+2b$.
36. Divide $x^4-a^2x^2+2a^3x-a^4$ by x^2-ax+a^2.

194.*a*. A regular series of quotients is obtained, by dividing the *difference* of the *powers* of two quantities, by the difference of the quantities or roots. Thus,

37. Divide (y^2-a^2) by $(y-a)$. Ans. $y+a$.
38. Divide (y^3-a^3) by $(y-a)$.
39. Divide (y^4-a^4) by $(y-a)$.
40. Divide (y^5-a^5) by $(y-a)$.

GREATEST COMMON MEASURE.

195. (1.) A *common measure* of two or more quantities, is a quantity which will *divide* or *measure* them without a remainder. (Art. 30.) Thus $2d$ is a common measure of $12d$, $6d$, $8d$, &c.

(2.) *The greatest common measure* of two or more quantities, is the greatest quantity which will divide these quantities without a remainder. Thus $6d$ is the greatest common measure of $12d$ and $18d$; and 8 is the greatest common measure of 16, 24 and 32.

195.*a*. To find the greatest common measure of two or more quantities.

Divide one of the quantities by the other, and the preceding divisor by the last remainder, till nothing remains; the last divisor will be the greatest common measure.

196. *The greatest common measure of two quantities is not altered by multiplying or dividing either of them by any quantity which is not a divisor of the other, and which contains no factor which is a divisor of the other.*

The common measure of ab and ac, is a. If either be *multiplied* by d, the common measure of abd, and ac, or of ab and acd, is still a. On the other hand, if ab and acd are

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QUEST.—What is a common measure? What the greatest common measure? How found? How is it affected by multiplying or dividing either of the quantities by any quantity which is not a divisor of the other?

the given quantities, the common measure is a; and if acd be *divided* by d, the common measure of ab and ac, is a.

Hence, in finding the common measure by division, the divisor may often be rendered more simple, by dividing it by some quantity which does not contain a divisor of the dividend. Or the dividend may be *multiplied* by a factor, which does not contain a measure of the divisor.

1. Find the greatest common measure of $6a^2+11ax+3x^2$ and $6a^2+7ax-3x^2$.

$$6a^2+7ax-3x^2)6a^2+11ax+3x^2(1$$
$$6a^2+7ax-3x^2$$

$$\text{Dividing by } 2x)4ax+6x^2$$

$$2a+3x)6a^2+7ax-3x^2(3a-x$$
$$6a^2+9ax$$
$$-2ax-3x^2$$
$$-2ax-3x^2$$
$$* \quad *$$

After the first division, the remainder is divided by $2x$, which reduces it to $2a+3x$. The division of the preceding divisor by this, leaves no remainder. Therefore $2a+3x$ is the common measure required.

2. What is the greatest common measure of x^3-b^2x and $x^2+2bx+b^2$?
3. What of $cx+x^2$ and a^2c+a^2x?
4. What of $3x^3-24x-9$ and $2x^3-16x-6$?
5. What of a^4-b^4 and $a^5-b^2a^3$?
6. What of x^2-1 and $xy+y$?
7. What of x^3-a^3 and x^4-a^4?
8. What of $a^2-ab-2b^2$ and $a^2-3ab+2b^2$?

9. What of a^4-x^4 and $a^3-a^2x-ax^2+x^3$?

10. What of a^3-ab^2 and $a^2+2ab+b^2$?

FRACTIONS CONTAINING POWERS.

197. In the section on fractions, the following examples were omitted for the sake of avoiding the anticipation of powers.

1. Reduce $\frac{5a^4}{3a^2}$ to lower terms. Ans. $\frac{5a^2}{3}$.

For $\frac{5a^4}{3a^2}=\frac{5aaaa}{3aa}=\frac{5aa}{3}$. (Art. 117.)

2. Reduce $\frac{6x^6}{3x^5}$.

3. Reduce $\frac{3a^4+4a^6}{5a^3}$.

4. Reduce $\frac{8a^3y-12a^2y^2+6ay^3}{6a^2y+4ay^2}$.

5. Reduce $\frac{a^2}{a^3}$ and $\frac{a^{-3}}{a^{-4}}$ to a common denominator.

$a^2\times a^{-4}$ is a^{-2}, the first numerator. (Art. 118.)
$a^3\times a^{-3}$ is $a^0=1$, the second numerator.
$a^3\times a^{-4}$ is a^{-1}, the common denominator.

The fractions reduced are therefore $\frac{a^{-2}}{a^{-1}}$ and $\frac{1}{a^{-1}}$.

6. Reduce $\frac{2a^4}{5a^3}$ and $\frac{a^2}{a^4}$ to a common denominator.

7. Multiply $\frac{3x^2}{4x^3}$ into $\frac{dx}{2x^4}$.

8. Multiply $\frac{a^3+b}{b^4}$ into $\frac{a-b}{3}$.

9. Multiply $\frac{a^5+1}{x^2}$ into $\frac{b^2-1}{x+a}$.

10. Multiply $\frac{b^4}{a^{-2}}$ into $\frac{h^{-3}}{x}$ and $\frac{a^n}{y^{-3}}$.

11. Divide $\frac{x^4}{y^3}$ by $\frac{a^3}{y^2}$.

12. Divide $\frac{a^3-x^4}{a^2}$ by $\frac{a^2-a^{-2}}{a}$.

13. Divide $\frac{b-y^{-1}}{y}$ by $\frac{a^3+b^{-4}}{y^3}$.

14. Divide $\frac{h^3-1}{d^4}$ by $\frac{d^n+1}{h}$.

SECTION IX.

ROOTS.

Art. 198. If we resolve b^3, or bbb, into equal factors, viz. b, b and b, each of these equal factors is said to be a root of b^3. So if we resolve 27 into any number of equal factors, as $3\times3\times3$, each of these equal factors is said to be a root of 27. And when any quantity is resolved into any number of equal factors, each of those factors is said to be a root of that quantity.

199. *A root of a quantity, then, is a factor, which, multiplied into itself a certain number of times, will produce that quantity.*

The number of times the root must be taken as a factor to produce the given quantity, is denoted by the name of the root.

Quest.—What is a root?

Thus 2 is the 4th root of 16; because $2\times2\times2\times2=16$, where two is taken *four* times as a factor to produce 16.

So a^3 is the square root of a^6; for $a^3\times a^3=a^6$. (Art. 189.)

Powers and roots are correlative terms. If one quantity is a power of another, the latter is a root of the former. As b^3 is the cube of b, so b is the cube root of b^3.

200. There are two methods in use, for expressing the roots of quantities; one by means of the radical sign $\sqrt{}$, and the other by a fractional index. The latter is generally to be preferred; but the former has its uses on particular occasions.

When a root is expressed by the radical sign, the sign is placed before the given quantity, in this manner, $\sqrt{a}$.

Thus $\sqrt[2]{a}$ is the 2d, or square root of a.

$\sqrt[3]{a}$ is the 3d, or cube root.

201. The *figure* placed over the radical sign, denotes the *number of factors*, into which the given quantity is resolved; i. e. the *number* of *times* the *root* must be taken as a *factor* to *produce* the *given quantity*.

Thus $\sqrt[2]{a^2}$ shows that a^2 is to be resolved into two factors, and $\sqrt[3]{a^3}$, into three factors; and $\sqrt[n]{a}$, into n factors.

The figure for the *square* root is commonly omitted, and the radical sign is simply written before the quantity, thus $\sqrt{a^2}=\sqrt[2]{a^2}$.

202. When a figure or letter is *prefixed* to the radical sign, without any character between them, the two quantities are to be considered as *multiplied* together.

QUEST.—How many ways to express the roots of quantities? The first? Second? Which is preferred? What does the figure placed over the radical sign denote? Is the figure used in denoting the square root? When a figure or letter is prefixed to the radical sign, what does it show?

Thus $2\sqrt{a}$, is $2\times\sqrt{a}$, that is, 2 multiplied into the root of a, or which is the same thing, *twice* the root of a.

And $x\sqrt{b}$, is $x\times\sqrt{b}$, or x times the root of b.

When no co-efficient is prefixed to the radical sign, 1 is always understood; $\sqrt{a}$ being the same as $1\sqrt{a}$, that is, *once* the root of a.

203. The *cube* root of a^6 is a^2. For $a^2\times a^2\times a^2=a^6$. (Art. 199.)

Here the index is divided into *three* equal parts, and the quantity itself resolved into three equal factors.

The square root of a^2 is a^1 or a. For $a\times a=a^2$.

By extending the same plan of notation, *fractional indices* are obtained.

Thus, in taking the square root of a^1 or a, the index 1 is divided into two equal parts, $\frac{1}{2}$ and $\frac{1}{2}$; and the root is $a^{\frac{1}{2}}$.

On the same principle, the cube root of a, is $a^{\frac{1}{3}}=\sqrt[3]{a}$.

The nth root, is $a^{\frac{1}{n}}=\sqrt[n]{a}$, &c.

204. Every root, as well as every power of 1, is 1. (Art. 165.) For a root is a factor, which multiplied into itself will produce the given quantity. But no factor except 1 can produce 1, by being multiplied into itself.

So that 1^n, 1, $\sqrt{1}$, $\sqrt[n]{1}$, &c., are all equal.

205. *Negative* indices are used in the notation of roots, as well as of powers. (See Art. 163, 4.)

Thus $\dfrac{1}{a^{\frac{1}{2}}}=a^{-\frac{1}{2}}$; $\dfrac{1}{a^{\frac{1}{3}}}=a^{-\frac{1}{3}}$; $\dfrac{1}{a^{\frac{1}{n}}}=a^{-\frac{1}{n}}$.

QUEST.—When none is prefixed, what is understood? What is every root of 1? Do roots ever have negative indices?

POWERS OF ROOTS.

206. In the preceding examples of roots, the numerator of the fractional index has been a *unit*. There is another class of quantities, the numerators of whose indices are *greater* than 1, as $b^{\frac{2}{3}}$, $c^{\frac{3}{4}}$, &c. These quantities may be considered either as *powers* of *roots*, or *roots* of *powers*.

N. B. In all instances, when the root of a quantity is denoted by a fractional index, the *denominator*, like the figure over the radical sign, (Art. 201,) expresses the *root*, and the *numerator* the *power*. Thus $a^{\frac{1}{3}}$ denotes the *cube* root of the *first* power of a, i. e. that a is to be resolved into three *equal factors*; for $a^{\frac{1}{3}}\times a^{\frac{1}{3}}\times a^{\frac{1}{3}}=a$. On the other hand, $c^{\frac{3}{4}}$ denotes the *third* power of the *fourth* root of c, or the *fourth* root of the *third* power. One expression is *equivalent* to the other.

1. What is $a^{\frac{4}{3}}$ equal to? 2. What is $x^{\frac{3}{2}}$ equal to?
3. What is $y^{\frac{8}{6}}$ equal to? 4. What is $b^{\frac{7}{8}}$ equal to?
5. Write the fifth root of the fourth power of a.
6. Write the seventh power of the ninth root of d.

207. The value of a quantity is not altered, by applying to it a fractional index whose numerator and denominator are equal.

Thus $a=a^{\frac{2}{2}}=a^{\frac{3}{3}}=a^{\frac{n}{n}}$. For the denominator shows that a is resolved into a certain number of factors; and the numerator shows that all these factors are included in $a^{\frac{n}{n}}$.

QUEST.—What is meant by powers of roots? What does the denominator of a fractional index express? What the numerator? Explain $x^{\frac{3}{4}}$; also $b^{\frac{2}{3}}$, $c^{\frac{9}{10}}$, $y^{\frac{20}{100}}$. When the numerator and denominator are equal, how does the index affect the quantity? How simplify such an expression?

On the other hand, when the numerator of a fractional index becomes equal to the denominator, the expression may be rendered more simple by *rejecting* the index.

Instead of $a^{\frac{n}{n}}$, we may write a.

207.*a*. The index of a power or root may be exchanged for any other index of the same value.

Instead of $a^{\frac{2}{3}}$, we may put $a^{\frac{4}{6}}$.

For in the latter of these expressions, a is supposed to be resolved into *twice* as many factors as in the former; and the numerator shows that *twice* as many of these factors are to be multiplied together. Hence the value is not altered.

208. From the preceding article, it will be easily seen, that a fractional index may be expressed in *decimals*.

7. Thus $a^{\frac{1}{2}}=a^{\frac{5}{10}}$, or $a^{0.5}$; that is, the square root is equal to the fifth power of the tenth root.

8. Express $a^{\frac{1}{4}}$ in decimals. 9. Express $a^{\frac{2}{5}}$ in decimals.

10. Express $a^{\frac{7}{2}}$ in decimals. 11. Express $a^{\frac{9}{5}}$ in decimals.

12. Express $a^{\frac{11}{4}}$ in decimals.

In many cases, however, the decimal can be only an *approximation* to the true index.

13. Thus $a^{\frac{1}{3}}=a^{0.3}$ nearly, or $a^{0.33333}$ more nearly.

In this manner, the approximation may be carried to any degree of exactness which is required.

14. Express $a^{\frac{5}{3}}$ in decimals. 15. Express $a^{\frac{11}{7}}$ in dec.

N. B. These decimal indices form a very important class of numbers, called *logarithms*.

QUEST.—What is the effect when one index is exchanged for another index of the same value? Can a fractional index be expressed in decimals? Can it be expressed exactly by decimals in all cases? What class of numbers are thus found?

EVOLUTION.

209. The process of *resolving* quantities into *equal factors*, is called *Evolution.*

In *subtraction*, a quantity is resolved into *two parts.*

In *division*, a quantity is resolved into *two factors.*

In *evolution*, a quantity is resolved into *equal factors.*

Evolution is the opposite of involution: One is finding a *power* of a quantity, by multiplying it into itself. The other is finding a *root*, by resolving a quantity into equal factors. A quantity is resolved into any number of equal factors, by dividing its *index* into as many *equal parts.*

210. From the foregoing principles we deduce the following

GENERAL RULE FOR EVOLUTION.

I. *Divide the index of the quantity by the number expressing the root to be found.* Or,

Place the radical sign belonging to the required root over the given quantity.

II. If the quantities have co-efficients, *the root of these must be extracted and placed before the radical sign or quantity.* Thus,

To find the square root of d^4, divide the index 4 by 2, i. e. $d^{\frac{4}{2}}=d^2$. So the cube root of d^6, is $d^{\frac{6}{3}}=d^2$.

Obser.—From the manner of performing evolution it is evident, that the plan of denoting roots by *fractional* indices, is derived from the mode of expressing powers by *integral* indices. (Art. 203.)

QUEST.—What is evolution? Into what are quantities resolved in subtraction? Into what, in division? Into what, in evolution? How is a quantity resolved into any number of equal factors? Rule for evolution? What is the plan of denoting roots by fractional indices derived from?

1. Required the cube root of a^6. Ans. a^2.

2. Required the cube root of a or a^1. Ans. $a^{\frac{1}{3}}$, or $\sqrt[3]{a}$.

For $a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}}$, or $\sqrt[3]{a} \times \sqrt[3]{a} \times \sqrt[3]{a} = a$. (Art. 199.)

3. Required the fifth root of ab.
4. Required the nth root of a^2.
5. Required the seventh root of $2d-x$.
6. Required the fifth root of $(a-x)^2$.
7. Required the cube root of $a^{\frac{1}{2}}$.
8. Required the fourth root of a^{-1}.
9. Required the cube root of $a^{\frac{2}{3}}$.
10. Required the nth root of x^m.
11. Required the third root of a^6.
12. Required the fourth root of x^8.
13. Required the second root of x^n.
14. Required the fifth root of d^3.
15. Required the 8th root of a^3.

210.*a*. The rule in the preceding article may be applied to every case in evolution. But when the quantity whose root is to be found, is composed of *several factors*, there will frequently be an advantage in taking the root of each of the factors *separately*.

This is done upon the principle that the *root of the product of several factors, is equal to the product of their roots.*

Thus $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$. For each member of the equation if involved, will give the same power.

When, therefore, a quantity consists of several factors, we may either extract the root of the whole together; or we may

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QUEST.—What is the root of the product of several factors equal to?

find the root of the factors separately, and then multiply them into each other.

16. The cube root of xy, is either $(xy)^{\frac{1}{3}}$, or $x^{\frac{1}{3}}y^{\frac{1}{3}}$.

17. Required the fifth root of $3y$.

18. Required the sixth root of abh.

19. Required the cube root of $8b$.

20. Required the nth root of $x^n y$.

211. *The root of a fraction is equal to the root of the numerator divided by the root of the denominator.*

21. Thus the square root of $\frac{a}{b} = \frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}$. For $\frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}} \times \frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}} = \frac{a}{b}$.

22. Required the nth root of $\frac{a}{b}$.

23. Required the square root of $\frac{x}{ay}$.

212. Signs.—(1.) *An odd root of any quantity has the same sign as the quantity itself.*

(2.) *An even root of an affirmative quantity is ambiguous.*

(3.) *An even root of a negative quantity is impossible.*

213. But an *even* root of an *affirmative* quantity may be either *positive* or *negative*. For, the quantity may be produced from the one, as well as from the other. (Art. 169.)

Thus the square root of a^2 is $+a$, or $-a$.

An even root of an affirmative quantity is, therefore, said to be *ambiguous*, and is marked with the sign $\pm$. Thus the square root of $3b$, is $\pm\sqrt{3b}$. The 4th root of x, is $\pm x^{\frac{1}{4}}$.

The ambiguity does not exist, however, when from the nature of the case, or a previous multiplication, it is known

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Quest.—What is the root of a fraction equal to? Rule for signs? What is the even root of a positive quantity? Of a negative?

whether the power has actually been produced from a *positive* or from a *negative* quantity.

214. But no *even* root of a *negative* quantity can be found. The square root of $-a^2$ is neither $+a$ nor $-a$.

For $+a\times+a=+a^2$. And $-a\times-a=+a^2$ also.

An *even* root of a *negative* quantity is, therefore, said to be *impossible* or *imaginary*. End

215. The methods of extracting the roots of *compound* quantities are to be considered in a future section. But there is one class of them, the squares of *binomial* and *residual* quantities, which it will be proper to attend to in this place. The square of $a+b$, for instance, is $a^2+2ab+b^2$, two terms of which, a^2 and b^2, are complete powers, and $2ab$ is twice the product of a into b, that is, the root of a^2 into the root of b^2. Whenever, therefore, we meet with a quantity of this description, we may know that its square root is a binomial; and this may be found, by taking the root of the two terms which are complete powers, and connecting them by the sign $+$. The other term disappears in the root. Thus, to find the square root of $x^2+2xy+y^2$, take the root of x^2, and the root of y^2, and connect them by the sign $+$. The binomial root will then be $x+y$.

In a *residual* quantity, the double product has the sign $-$ prefixed, instead of $+$. The square of $a-b$, for instance, is $a^2-2ab+b^2$. (Art. 173.) And to obtain the root of a quantity of this description, we have only to take the roots of the two complete powers, and connect them by the sign $-$. Thus the square root of $x^2-2xy+y^2$, is $x-y$. Hence,

216. To extract the square root of a *binomial* or *residual*.

Take the roots of the two terms which are complete powers, and connect them by the sign which is prefixed to the other term.

QUEST.—How extract the square root of a binomial or residual?

1. To find the root of x^2+2x+1.

The two terms which are complete powers, are x^2 and 1. The roots are x and 1. (Art. 204.) Then $x+1$. Ans.

2. Find the square root of x^2-2x+1. (Art. 173.)

3. Find the square root of $a^2+a+\frac{1}{4}$.

4. Find the square root of $a^2+\frac{4}{3}a+\frac{4}{9}$.

5. Find the square root of $a^2+ab+\frac{b^2}{4}$.

6. Find the square root of $a^2+\frac{2\,ab}{c}+\frac{b^2}{c^2}$.

217. *A root whose value cannot be exactly expressed in numbers, is called a* SURD, *or irrational quantity.*

Thus $\sqrt{2}$ is a surd, because the square root of 2 cannot be expressed in numbers, with perfect exactness.

In decimals, it is 1.41421356 nearly.

218. Every quantity which is not a *surd*, is said to be *rational.*

219. By Radical Quantities is meant, all quantities which are found under the *radical sign*, or *which have a fractional index.*

REDUCTION OF RADICAL QUANTITIES.

220. Case I. To reduce a *rational* quantity to the form of a *radical* without altering its value.

Raise the quantity to a power of the same name as the given root, and then apply the corresponding radical sign or index.

Quest.—What is a surd? What a rational quantity? What are radical quantities? How reduce a *rational* quantity to the *form* of a radical?

1. Reduce a to the form of the nth root.

The nth power of a is a^n. (Art. 166.)

Over this, place the radical sign, and it becomes $\sqrt[n]{a^n}$.

It is thus reduced to the form of a radical quantity, without any alteration of its value. For $\sqrt[n]{a^n}=a^{\frac{n}{n}}=a$. (Art. 207.)

2. Reduce 4 to the form of the cube root.
3. Reduce $3a$ to the form of the 4th root.
4. Reduce $\frac{1}{3}ab$ to the form of the square root.
5. Reduce $3\times(a-x)$ to the form of the cube root
6. Reduce a^3 to the form of the cube root.

N. B. In cases of this kind, where a *power* is to be reduced to the form of the nth root, it must be raised to the nth power, not of the *given letter*, but of the *power* of the letter.

Thus in the 6th example, a^6 is the cube, not of a, but of a^2.

7. Reduce a^3b^4 to the form of the square root.
8. Reduce a^m to the form of the nth root.

221. Case II. To reduce quantities which have *different indices*, to others of the same value having a *common index*.

(1.) *Reduce the indices to a common denominator.*

(2.) *Involve each quantity to the power expressed by the numerator of the reduced index.*

(3.) *Take the root denoted by the common denominator.*

9. Reduce $a^{\frac{1}{4}}$ and $b^{\frac{1}{6}}$ to a common index.

1st. The indices $\frac{1}{4}$ and $\frac{1}{6}$ reduced to a common denominator, are $\frac{3}{12}$ and $\frac{2}{12}$. (Art. 118.)

2d. The quantities a and b involved to the powers expressed by the two numerators, are a^3 and b^2.

Quest.—How reduce quantities which have *different* indices to a *common* index?

3d. The root denoted by the common denominator is the $\frac{1}{12}$th. The answer, then, is $(a^3)^{\frac{1}{12}}$ and $(b^8)^{\frac{1}{12}}$.

The two quantities are thus reduced to a common index, without any alteration in their values.

For by Art. 207.*a*, $a^{\frac{1}{4}}=a^{\frac{3}{12}}$, which by Art. 206, $=(a^3)^{\frac{1}{12}}$.

And universally, $a^{\frac{1}{n}}=a^{\frac{m}{mn}}=(a^m)^{\frac{1}{mn}}$.

10. Reduce $a^{\frac{1}{2}}$ and $bx^{\frac{2}{3}}$ to a common index.

Ans. $a^{\frac{3}{6}}$ and $(bx)^{\frac{4}{6}}$, or $(a^3)^{\frac{1}{6}}$ and $(b^4x^4)^{\frac{1}{6}}$.

11. Reduce a^2 and $b^{\frac{1}{n}}$.

12. Reduce $x^{\frac{1}{n}}$ and $y^{\frac{1}{m}}$.

13. Reduce $2^{\frac{1}{2}}$ and $3^{\frac{1}{3}}$.

14. Reduce $(a+b)^2$ and $(x-y)^{\frac{2}{3}}$.

15. Reduce $a^{\frac{1}{3}}$ and $b^{\frac{1}{5}}$.

16. Reduce $x^{\frac{2}{3}}$ and $5^{\frac{1}{2}}$.

222. Case III. To reduce a quantity to a *given index.*

Divide the index of the quantity by the given index, place the quotient over the quantity, and set the given index over the whole.

This is merely resolving the original index into two factors. (Art. 209.)

17. Reduce $a^{\frac{1}{6}}$ to the index $\frac{1}{2}$.

By Art. 135, $\frac{1}{6}\div\frac{1}{2}=\frac{1}{6}\times\frac{2}{1}=\frac{2}{6}=\frac{1}{3}$.

This is the index to be placed over *a*, which then becomes $a^{\frac{1}{3}}$; and the given index set over this, makes it $(a^{\frac{1}{3}})^{\frac{1}{2}}$, the answer.

18. Reduce a^2 and $x^{\frac{3}{2}}$ to the common index $\frac{1}{3}$.

$$\left.\begin{array}{l} 2\div\frac{1}{3}=2\times3=6\text{, the first index.} \\ \frac{3}{2}\div\frac{1}{3}=\frac{3}{2}\times3=\frac{9}{2}\text{, the second index.}\end{array}\right\}$$

Therefore $(a^6)^{\frac{1}{3}}$ and $(x^{\frac{9}{2}})^{\frac{1}{3}}$ are the quantities required.

Quest.—How reduce a quantity to a *given* index?

19. Reduce $4^{\frac{1}{2}}$ and $3^{\frac{1}{3}}$ to the common index $\frac{1}{6}$.

20. Reduce x^2 and y^4 to the common index $\frac{1}{2}$.

21. Reduce $a^{\frac{1}{2}}$ and $b^{\frac{1}{3}}$ to the common index $\frac{1}{6}$.

22. Reduce c^2 and $d^{\frac{1}{2}}$ to the common index $\frac{3}{4}$.

23. Reduce $a^{\frac{1}{n}}$ and $b^{\frac{2}{m}}$ to the common index $\frac{1}{n}$.

24. Reduce $a^{\frac{1}{2}}$, $b^{\frac{1}{3}}$ and $c^{\frac{1}{4}}$ to the common index $\frac{1}{12}$.

223. Case IV. To reduce a radical quantity to its *most simple terms*; i. e. to remove a factor from under the radical sign.

Resolve the quantity into two factors, one of which is an exact power of the same name with the root; find the root of this power, and prefix it to the other factor, with the radical sign between them.

This rule is founded on the principle, that the root of the *product* of two factors is equal to the product of their roots. (Art. 210.*a.*)

It will generally be best to resolve the radical quantity into such factors, that one of them shall be the *greatest* power which will divide the quantity without a remainder.

N. B. If there is no exact power which will divide the quantity, the reduction cannot be made.

25. Remove a factor from $\sqrt{8}$.

The greatest square which will divide 8 is 4.

We may then resolve 8 into the factors 4 and 2 For $4\times2=8$.

The root of this product is equal to the product of the roots of its factors; that is $\sqrt{8}=\sqrt{4}\times\sqrt{2}$.

But $\sqrt{4}=2$. Instead of $\sqrt{4}$, therefore, we may substitute its equal 2. We then have $2\times\sqrt{2}$, or $2\sqrt{2}$.

Quest.—How reduce a radical quantity to its simplest terms.

26. Reduce $\sqrt{a^2x}$. Ans. $\sqrt{a^2}\times\sqrt{x}=a\times\sqrt{x}=a\sqrt{x}$.

27. Reduce $\sqrt{18}$.

28. Reduce $\sqrt[3]{64b^3c}$.

29. Reduce $\sqrt[4]{\dfrac{a^4b}{c^5d}}$.

30. Reduce $\sqrt[n]{a^nb}$.

31. Reduce $(a^3-a^2b)^{\frac{1}{2}}$.

32. Reduce $(54a^6b)^{\frac{1}{3}}$.

33. Reduce $\sqrt{98a^2x}$.

34. Reduce $\sqrt[3]{a^3+a^3b^2}$.

224. Case V. To introduce a *co-efficient* of a radical quantity under the radical sign. (Art. 220.)

Raise the co-efficient to a power of the same name as the radical part, then place it as a factor under the radical sign.

35. Thus, $a\sqrt[n]{b}=\sqrt[n]{a^nb}$.

For $a=\sqrt[n]{a^n}$ or $a^{\frac{n}{n}}$. (Art. 207.) And $\sqrt[n]{a^n}\times\sqrt[n]{b}=\sqrt[n]{a^nb}$.

36. Reduce $a(x-b)^{\frac{1}{3}}$ to the form of a radical.

Ans. $(a^3x-a^3b)^{\frac{1}{3}}$.

37. Reduce $2ab(2ab^2)^{\frac{1}{3}}$.

38. Reduce $\dfrac{a}{b}\left(\dfrac{b^2c}{a^2+b^2}\right)^{\frac{1}{2}}$.

39. Reduce $2\sqrt{2}$.

40. Reduce $4b\sqrt[3]{c}$.

EXAMPLES FOR PRACTICE.

1. Reduce $5\sqrt{6}$ to a simple radical.
2. Reduce $\frac{1}{5}\sqrt{5a}$ to a simple radical.
3. Reduce $5^{\frac{1}{2}}$ and $6^{\frac{1}{3}}$ to the common index $\frac{1}{6}$.
4. Reduce a^2 and $a^{\frac{1}{2}}$ to the common index $\frac{1}{4}$.
5. Reduce $\sqrt{98}$ to its simplest form.
6. Reduce $\sqrt{243}$ to its simplest form.

Quest.—How introduce a co-efficient under a radical sign?

7. Reduce $\sqrt[3]{54}$ to its simplest form.
8. Reduce $7\sqrt{80}$ to its simplest form.
9. Reduce $9\sqrt[3]{81}$ to its simplest form.
10. Reduce $\sqrt{x^2+ax^2}$ to its simplest form. $=\sqrt{}$? $/+a$
11. Reduce $\sqrt{198a^2x}$ to its simplest form.
12. Reduce $\sqrt{x^3-a^2x^2}$ to its simplest form.

ADDITION OF RADICAL QUANTITIES.

225. It may be proper to remark, that the rules for addition, subtraction, multiplication and division of *radical quantities* depend on the *same principles*, and are expressed in nearly the same language, as those for addition, subtraction, multiplication and division of *powers*. So also the rules for involution and evolution of radicals, are *similar* to those for involution and evolution of powers. Hence, if the learner has made himself thoroughly acquainted with the principles and operations relating to powers, he has substantially acquired those pertaining to radical quantities, and will find no difficulty in understanding and applying them.

225.*a*. When radical quantities have the *same radical part*, and are under the *same radical sign* or *index*, they are *like* quantities. (Art. 28.) Hence their *rational parts* or *co-efficients* may be added in the same manner as *rational* quantities, (Art. 56,) and the sum prefixed to the *radical* part. Thus, $2\sqrt{b}+3\sqrt{b}=5\sqrt{b}$.

1. Add $\sqrt[n]{ay}$ to $2\sqrt[n]{ay}$.
2. Add $-2\sqrt{a}$ to $5\sqrt{a}$.
3. Add $4(x+h)^{\frac{1}{7}}$ to $3(x+h)^{\frac{1}{7}}$.

QUEST.—What is said respecting the rules for addition, subtraction, multiplication and division: also of involution and evolution of radicals? How are radical quantities added, when the radical parts are *alike?*

4. Add $7bh^{\frac{1}{6}}$ to $5bh^{\frac{1}{6}}$.

5. Add $y\sqrt{b-h}$ to $a\sqrt{b-h}$.

226. If the radical parts are originally different, they may sometimes be made alike, by the rules for reduction of radical quantities.

6. Add $\sqrt{8}$ to $\sqrt{50}$. Here the radical parts are not the same. But by reduction as in Art. 223, $\sqrt{8}=2\sqrt{2}$, and $\sqrt{50}=5\sqrt{2}$. And $2\sqrt{2}+5\sqrt{2}=7\sqrt{2}$. Ans.

7. Add $\sqrt{16b}$ to $\sqrt{4b}$.

8. Add $\sqrt{a^2x}$ to $\sqrt{b^4x}$.

9. Add $(36a^2y)^{\frac{1}{2}}$ to $(25y)^{\frac{1}{2}}$.

10. Add $\sqrt{18a}$ to $3\sqrt{2a}$.

227. But if the radical parts, after reduction, are *different*, or have *different exponents*, the quantities are *unlike*, (Art. 28;) hence they can be added only by writing them one after the other with their signs. (Art. 55.)

11. The sum of $3\sqrt{b}$ and $2\sqrt{a}$, is $3\sqrt{b}+2\sqrt{a}$.

It is manifest that three times the root of b, and twice the root of a, are neither five times the root of b, nor five times the root of a, unless b and a are equal.

12. The sum of $\sqrt[2]{a}$ and $\sqrt[3]{a}$, is $\sqrt[2]{a}+\sqrt[3]{a}$.

The *square* root of a, and the *cube* root of a, are neither twice the square root, nor twice the cube root of a.

228. From the preceding principle we deduce the following

GENERAL RULE FOR ADDITION OF RADICALS.

1. *If the radicals are like quantities, add their co-efficients, and to the sum annex the common radical parts.*

QUEST.—If they are originally different, how can they be made alike? When they are *unlike* quantities, how add them? General rule?

II. *If the radicals are unlike quantities, they must be added by writing them, one after another, without altering their signs.* (Art. 186.)

EXAMPLES FOR PRACTICE.

1. Add $\sqrt{27}$ to $\sqrt{48}$. (Art. 226.)
2. Add $\sqrt{72}$ to $\sqrt{128}$.
3. Add $\sqrt{180}$ to $\sqrt{405}$.
4. Add $3\sqrt[3]{40}$ to $\sqrt[3]{135}$.
5. Add $5\sqrt[3]{54}$ to $5\sqrt[3]{128}$.
6. Add $9\sqrt{243}$ to $10\sqrt{363}$.
7. Add $\sqrt{81b}$ to $\sqrt{49b}$.
8. Add $\sqrt{9a^2d}$ to $\sqrt{16a^2d}$.
9. Add $x\sqrt{25z^2c}$ to $\sqrt{36x^4c}$.
10. Add $3\sqrt[3]{a^6b}$ to $4a\sqrt[3]{a^3b}$.

SUBTRACTION OF RADICAL QUANTITIES.

229. Rule.—*Subtraction of radicals is performed in the same manner as addition, except that the signs of the subtrahend must be changed as in subtraction of powers.* (Art. 187.)

1. From $\sqrt{ay}$ take $3\sqrt{ay}$. Ans. $-2\sqrt{ay}$.
2. From $4\sqrt[n]{a+x}$ take $3\sqrt[n]{a+x}$.
3. From $3h^{\frac{1}{3}}$ take $-5h^{\frac{1}{3}}$.
4. From $a(x+y)^{\frac{1}{4}}$ take $b(x+y)^{\frac{1}{4}}$.
5. From $-a^{-\frac{1}{n}}$ take $-2a^{-\frac{1}{n}}$.
6. From $\sqrt{50}$ take $\sqrt{8}$.
7. From $\sqrt[3]{b^4y}$ take $\sqrt[3]{by^4}$.

Quest.—How are radical quantities subtracted?

8. From $\sqrt[4]{x}$ take $\sqrt[4]{x}$.
9. From $2\sqrt{50}$ take $\sqrt{18}$.
10. From $\sqrt[3]{320}$ take $\sqrt[3]{40}$.
11. From $5\sqrt{20}$ take $3\sqrt{45}$.
12. From $\sqrt{80a^4x}$ take $\sqrt{20a^2x}$.

MULTIPLICATION OF RADICAL QUANTITIES.

530. Radical quantities may be multiplied, like other quantities, by writing the factors one after another, either with, or without the sign of multiplication between them. (Art. 72.)

1. Thus the product of $\sqrt{a}$ into $\sqrt{b}$, is $\sqrt{a}\times\sqrt{b}$.

2. The product of $h^{\frac{1}{3}}$ into $y^{\frac{1}{2}}$, is $h^{\frac{1}{3}}y^{\frac{1}{2}}$.

But it is often expedient to bring the factors under the same radical sign. This may be done, if they are first reduced to a common index. (Art. 221.)

231. *Hence, quantities under the same radical sign or index may be multiplied together like rational quantities, the product being placed under the common radical sign or index.** (Art. 210.*a*.)

3. Multiply $\sqrt[2]{x}$ into $\sqrt[3]{y}$, that is, $x^{\frac{1}{2}}$ into $y^{\frac{1}{3}}$.

The quantities reduced to the same index, (Art. 221,) are $(x^3)^{\frac{1}{6}}$, and $(y^2)^{\frac{1}{6}}$, and their product is, $(x^3y^2)^{\frac{1}{6}}=\sqrt[6]{x^3y^2}$. Ans.

4. Multiply $\sqrt{a+m}$ into $\sqrt{a-m}$.

5. Multiply $\sqrt{dx}$ into $\sqrt{hy}$.

QUEST.—How may radical quantities be multiplied? How are factors brought under the same radical sign? How multiplied when under the same radical sign?

* The case of an *imaginary* root of a negative quantity may be considered an exception. (Art. 214.)

6. Multiply $a^{\frac{2}{3}}$ into $x^{\frac{1}{2}}$.

7. Multiply $(a+y)^{\frac{1}{n}}$ into $(b+h)^{\frac{1}{n}}$.

8. Multiply $a^{\frac{1}{m}}$ into $x^{\frac{1}{n}}$.

9. Multiply $\sqrt{8xb}$ into $\sqrt{2xb}$. Prod. $\sqrt{16x^2b^2}=4xb$

In this manner the product of radical quantities often becomes *rational.*

10. Thus the product of $\sqrt{2}$ into $\sqrt{18}=\sqrt{36}=6$. Ans.

11. Multiply $(a^2y^3)^{\frac{1}{4}}$ into $(a^2y)^{\frac{1}{4}}$.

232. *Roots of the same letter or quantity may be multiplied, by adding their fractional exponents.*

N. B. The exponents, like all other fractions, must be reduced to a common denominator, before they can be united in one term. (Art. 122.)

12. Thus $a^{\frac{1}{2}}\times a^{\frac{1}{3}}=a^{\frac{1}{2}+\frac{1}{3}}=a^{\frac{3}{6}+\frac{2}{6}}=a^{\frac{5}{6}}$.

The values of the roots are not altered by reducing their indices to a common denominator. (Art. 207.*a.*)

$$\left.\begin{array}{ll}\text{Therefore the first factor} & a^{\frac{1}{2}}=a^{\frac{3}{6}} \\ \text{And the second} & a^{\frac{1}{3}}=a^{\frac{2}{6}}\end{array}\right\}$$

But $a^{\frac{3}{6}}=a^{\frac{1}{6}}\times a^{\frac{1}{6}}\times a^{\frac{1}{6}}$. (Art. 206.)

And $a^{\frac{2}{6}}=a^{\frac{1}{6}}\times a^{\frac{1}{6}}$.

The product therefore is $a^{\frac{1}{6}}\times a^{\frac{1}{6}}\times a^{\frac{1}{6}}\times a^{\frac{1}{6}}\times {}^{\frac{1}{6}}=a^{\frac{5}{6}}$.

N. B. In all instances of this nature, the common denominator of the indices denotes a certain root; and the sum of

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QUEST.—How multiply roots of the same letter? How are exponents united?

12

the numerators, shows how often this is to be repeated as a factor to produce the required product.

13. Thus $a^{\frac{1}{n}} \times a^{\frac{1}{m}} = a^{\frac{m}{mn}} \times a^{\frac{n}{mn}} = a^{\frac{m+n}{mn}}$.

14. Multiply $3y^{\frac{1}{4}}$ into $y^{\frac{2}{3}}$.

15. Multiply $(a+b)^{\frac{1}{2}}$ into $(a+b)^{\frac{1}{4}}$.

16. Multiply $(a-y)^{\frac{1}{n}}$ into $(a-y)^{\frac{1}{m}}$.

17. Multiply $x^{-\frac{1}{4}}$ into $x^{-\frac{1}{3}}$.

18. Multiply $y^{\frac{1}{2}}$ into $y^{-\frac{1}{3}}$.

19. Multiply $a^{\frac{1}{n}}$ into $a^{-\frac{1}{n}}$.

20. Multiply $x^{n-\frac{1}{2}}$ into $x^{\frac{1}{2}-n}$.

21. Multiply a^2 into $a^{\frac{1}{3}}$.

233. Any quantities may be reduced to tne form of radicals, and may then be subjected to the same modes of operation. (Art. 220.)

22. Thus $y^3 \times y^{\frac{1}{6}} = y^{3+\frac{1}{6}} = y^{\frac{19}{6}}$. 23. And $x \times x^{\frac{1}{n}} = x^{\frac{n+1}{n}}$.

N. B. The product will become *rational*, whenever the *numerator* of the index can be exactly divided by the *denominator*.

24. Thus $a^3 \times a^{\frac{1}{3}} \times a^{\frac{2}{3}} = a^{\frac{12}{3}} = a^4$. (Art. 207.)

25. Multiply $(a+b)^{\frac{4}{3}}$ into $(a+b)^{-\frac{1}{3}}$.

26. Multiply $a^{\frac{3}{5}}$ into $a^{\frac{2}{5}}$.

234. When radical quantities which are reduced to the same index, have *rational co-efficients, the rational parts may*

Quest.—Can all quantities be reduced to the form of radicals? How may they be treated then? When radicals have co-efficients, what must be done with them?

be multiplied together, and their product prefixed to the product of the radical parts.

27. Multiply $a\sqrt{b}$ into $c\sqrt{d}$.

The product of the rational parts is ac.

The product of the radical parts is $\sqrt{bd}$.

And the whole product$=ac\sqrt{bd}$. Ans.

28. Multiply $ax^{\frac{1}{2}}$ into $bd^{\frac{1}{3}}$. Ans. $ab(x^3d^2)^{\frac{1}{6}}$.

But in cases of this nature we may save the trouble of reducing to a common index, by multiplying as in Art. 230.

29. Thus $ax^{\frac{1}{2}}$ into $bd^{\frac{1}{3}}=ax^{\frac{1}{2}}bd^{\frac{1}{3}}$. Ans.

30. Multiply $a(b+x)^{\frac{1}{2}}$ into $y(b-x)^{\frac{1}{2}}$.

31. Multiply $a\sqrt{y^2}$ into $b\sqrt{hy}$.

32. Multiply $a\sqrt{x}$ into $b\sqrt{x}$.

33. Multiply $ax^{-\frac{1}{2}}$ into $by^{-\frac{1}{2}}$.

34. Multiply $x\sqrt[3]{3}$ into $y\sqrt[3]{9}$.

235. If the rational quantities, instead of being *co-efficients* to the radical quantities, are connected with them by the signs $+$ and $-$, each term in the multiplier must be multiplied into each term in the multiplicand, as in Art. 78.

35. Multiply $a+\sqrt{b}$

Into $c+\sqrt{d}$

$$\begin{array}{l} ac+c\sqrt{b} \\ \qquad a\sqrt{d}+\sqrt{bd} \\ \hline ac+c\sqrt{b}+a\sqrt{d}+\sqrt{bd}. \text{ Ans.} \end{array}$$

36. Multiply $a+\sqrt{y}$ into $1+r\sqrt{y}$.

$a+\sqrt{y}+ar\sqrt{y}+ry$. Ans.

QUEST.—When the radicals are compound quantities, how proceed?

236. Hence we deduce the following

GENERAL RULE FOR MULTIPLYING RADICALS.

I. *Radicals of the same root, are multiplied by adding their fractional exponents.*

II. *If the quantities have the same radical sign, or index, multiply them together as you multiply rational quantities, place the product under the common radical sign, and to this prefix the product of their co-efficients.*

III. *If the radicals are compound quantities, each term in the multiplier must be multiplied into each term of the multiplicand by writing the terms one after another, either with, or without the sign of multiplication between them.* (Art. 189.)

EXAMPLES FOR PRACTICE.

1. Multiply $\sqrt{a}$ into $\sqrt[3]{b}$.
2. Multiply $5\sqrt{5}$ into $3\sqrt{8}$.
3. Multiply $2\sqrt{3}$ into $3\sqrt[3]{4}$.
4. Multiply $\sqrt{d}$ into $\sqrt[3]{ab}$.
5. Multiply $\sqrt{\frac{2ab}{3c}}$ into $\sqrt{\frac{9ad}{2b}}$.
6. Multiply $a(a-x)^{\frac{1}{2}}$ into $(c-d)\times(ax)^{\frac{1}{2}}$.
7. Multiply $5\sqrt{8}$ into $3\sqrt{5}$.
8. Multiply $\frac{1}{4}\sqrt{6}$ into $\frac{2}{15}\sqrt{9}$.
9. Multiply $\frac{1}{2}\sqrt{18}$ into $5\sqrt{20}$.
10. Multiply $2\sqrt{3}$ into $13\frac{1}{2}\sqrt{5}$.
11. Multiply $72\frac{1}{4}a^{\frac{2}{3}}$ into $120\frac{1}{2}a^{\frac{1}{4}}$.
12. Multiply $4+2\sqrt{2}$ into $2-\sqrt{2}$.

QUEST.—General rule for multiplying radical quantities?

DIVISION OF RADICAL QUANTITIES.

237. The division of radical quantities may be expressed by writing the divisor under the dividend, in the form of a fraction.

1. Thus the quotient of $\sqrt[3]{a}$ divided by $\sqrt{b}$, is $\frac{\sqrt[3]{a}}{\sqrt{b}}$.

2. And $(a+h)^{\frac{1}{3}}$ divided by $(b+x)^{\frac{1}{n}}$ is $\frac{(a+h)^{\frac{1}{3}}}{(b+x)^{\frac{1}{n}}}$.

In these instances, the radical sign or index is *separately* applied to the numerator and denominator. But if the divisor and dividend are reduced to the *same* index or radical sign, this may be applied to the *whole* quotient.

3. Thus $\sqrt[n]{a}\div\sqrt[n]{b}=\frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\sqrt[n]{\frac{a}{b}}$. For the root of a fraction is equal to the root of the numerator divided by the root of the denominator. (Art. 211.)

4. Again, $\sqrt[n]{ab}\div\sqrt[n]{b}=\sqrt[n]{a}$. For the product of this quotient into the divisor is equal to the dividend; that is,

$$\sqrt[n]{a}\times\sqrt[n]{b}=\sqrt[n]{ab}.$$ Hence,

238. *Quantities under the same radical sign or index, may be divided like rational quantities, the quotient being placed under the common radical sign or index.*

5. Divide $(x^3y^2)^{\frac{1}{6}}$ by $y^{\frac{1}{3}}$.

These reduced to the same index are $(x^3y^2)^{\frac{1}{6}}$ and $(y^2)^{\frac{1}{6}}$.

And the quotient is $(x^3)^{\frac{1}{6}}=x^{\frac{3}{6}}=x^{\frac{1}{2}}$. Ans.

QUEST.—How is the division of radicals expressed? How is the radical sign to be placed in this case? How divide quantities under the same radical sign?

6. Divide $\sqrt{6a^3x}$ by $\sqrt{3x}$.

7. Divide $\sqrt{dhx^2}$ by $\sqrt{dx}$.

8. Divide $(a^3+ax)^{\frac{1}{3}}$ by $a^{\frac{1}{3}}$.

9. Divide $(a^3h)^{\frac{1}{m}}$ by $(ax)^{\frac{1}{m}}$.

10. Divide $(a^2y^2)^{\frac{1}{4}}$ by $(ay)^{\frac{1}{4}}$.

239. *A root is divided by another root of the same letter or quantity, by subtracting the index of the divisor from that of the dividend.*

11. Thus $a^{\frac{1}{2}} \div a^{\frac{1}{6}} = a^{\frac{1}{2}-\frac{1}{6}} = a^{\frac{3}{6}-\frac{1}{6}} = a^{\frac{2}{6}} = a^{\frac{1}{3}}$,

For $a^{\frac{1}{2}} = a^{\frac{3}{6}} = a^{\frac{1}{6}} \times a^{\frac{1}{6}} \times a^{\frac{1}{6}}$, and this divided by $a^{\frac{1}{6}}$ is

$$\frac{a^{\frac{1}{6}} \times a^{\frac{1}{6}} \times a^{\frac{1}{6}}}{a^{\frac{1}{6}}} = a^{\frac{1}{6}} \times a^{\frac{1}{6}} = x^{\frac{2}{6}} = a^{\frac{1}{3}}.$$

12. In the same manner, $a^{\frac{1}{m}} \div a^{\frac{1}{n}} = a^{\frac{1}{m}-\frac{1}{n}}$.

13. Divide $(3a)^{\frac{1}{12}}$ by $a^{\frac{2}{3}}$.

14. Divide $(ax)^{\frac{2}{3}}$ by $(ax)^{\frac{1}{3}}$.

15. Divide $a^{\frac{m+n}{nm}}$ by $a^{\frac{1}{m}}$.

16. Divide $(b+y)^{\frac{2}{n}}$ by $(b+y)^{\frac{1}{n}}$.

17. Divide $(r^2y^3)^{\frac{1}{7}}$ by $(r^2y)^{3\frac{3}{7}}$.

239 *a*. *Powers* and *roots* of the same letter, may also be divided by each other, according to the preceding article.

18. Thus $a^2 \div a^{\frac{1}{3}} = a^{2-\frac{1}{3}} = a^{\frac{5}{3}}$. For $a^{\frac{5}{3}} \times a^{\frac{1}{3}} = a^{\frac{6}{3}} = a^2$.

QUEST.—How divide one root by another root of the same letter? How powers and roots of the same letter?

240. *When radical quantities which are reduced to the same index, have rational co-efficients, the rational parts may be divided separately, and their quotient prefixed to the quotient of the radical parts.*

19. Thus $ac\sqrt{bd}\div a\sqrt{b}=c\sqrt{d}$. For this quotient multiplied into the divisor is equal to the dividend.

20. Divide $24x\sqrt{ay}$ by $6\sqrt{a}$.

21. Divide $18dh\sqrt{bx}$ by $2h\sqrt{x}$.

22. Divide $by(a^3x^2)^{\frac{1}{n}}$ by $y(ax)^{\frac{1}{n}}$.

23. Divide $16\sqrt{32}$ by $8\sqrt{4}$.

24. Divide $b\sqrt{xy}$ by $\sqrt{y}$.

25. Divide $ab(x^2b)^{\frac{1}{4}}$ by $a(x)^{\frac{1}{2}}$.

These reduced to the same index are $ab(x^2b)^{\frac{1}{4}}$ and $a(x^2)^{\frac{1}{4}}$.

The quotient then is $b(b)^{\frac{1}{4}}=(b^5)^{\frac{1}{4}}$. (Art. 224.)

To save the trouble of reducing to a common index, the division may be expressed in the form of a fraction.

The quotient will then be $\dfrac{ab(x^2b)^{\frac{1}{4}}}{a(x)^{\frac{1}{2}}}$.

241. Hence we deduce the following

GENERAL RULE FOR DIVIDING RADICALS.

I. *If the radicals consist of the same letter or quantity, subtract the index of the divisor from that of the dividend, and place the remainder over the common radical part or root.*

II. *If the radicals have co-efficients, the co-efficient of the dividend must be divided by that of the divisor.* (Art. 96.)

QUEST.—When the radicals have co-efficients, what is to be done with them? General rule for dividing radical quantities?

III. *If the quantities have the same radical sign or index, divide them as rational quantities, and place the quotient under the common radical sign.* (Art. 193.)

EXAMPLES FOR PRACTICE

1. Divide $2\sqrt[3]{bc}$ by $3\sqrt{ac}$.
2. Divide $10\sqrt[3]{108}$ by $5\sqrt[3]{4}$.
3. Divide $10\sqrt{27}$ by $2\sqrt{3}$.
4. Divide $8\sqrt{108}$ by $2\sqrt{6}$.
5. Divide $(a^2b^2d^3)^{\frac{1}{6}}$ by $d^{\frac{1}{2}}$.
6. Divide $(16a^3-12a^2x)^{\frac{1}{2}}$ by $2a$.
7. Divide $6\sqrt{138}$ by $2\sqrt{6}$.
8. Divide $8\sqrt[3]{512}$ by $4\sqrt[3]{2}$.
9. Divide $\frac{1}{2}\sqrt{5}$ by $\frac{1}{3}\sqrt{2}$.
10. Divide $\sqrt{7}$ by $\sqrt[3]{7}$.
11. Divide $6\sqrt{54}$ by $3\sqrt{2}$.
12. Divide $4\sqrt[3]{72}$ by $2\sqrt[3]{18}$.

INVOLUTION OF RADICAL QUANTITIES.

242. To involve a radical quantity to any required power.

Multiply the index of the root into the index of the power to which it is to be raised. (Art. 170.)

1. Thus the square of $a^{\frac{1}{3}}=a^{\frac{1}{3}\times 2}=a^{\frac{2}{3}}$. For $a^{\frac{1}{3}}\times a^{\frac{1}{3}}=a^{\frac{2}{3}}$.

2. Required the cube of $a^{\frac{1}{4}}$.

3. Required the nth power of $a^{\frac{1}{m}}$.

4. Required the fifth power of $a^{\frac{1}{2}}y^{\frac{1}{3}}$.

QUEST.—How are radical quantities involved?

5. Required the cube of $a^{\frac{1}{n}} x^{\frac{1}{m}}$.

6. Required the square of $a^{\frac{2}{3}}x^{\frac{3}{4}}$.

7. Required the cube of $a^{\frac{1}{3}}$.

8. Required the *n*th power of $a^{\frac{1}{n}}$.

243. *A root is raised to a power of the same name, by removing the index or radical sign.*

N. B. When the radical quantities have *rational co-efficients*, these must be involved by actual multiplication.

9. Thus the cube of $\sqrt[3]{b+x}$, is $b+x$.

10. And the *n*th power of $(a-y)^{\frac{1}{n}}$, is $(a-y.)$

11. The square of $a\sqrt[n]{x}$, is $a^2\sqrt[n]{x^2}$.
For $a\sqrt[n]{x}\times a\sqrt[n]{x}=a^2\sqrt[n]{x2}$.

12. Required the *n*th power of $a^m x^m$.

13. Required the square of $a\sqrt{x-y}$.

14. Required the cube of $3a\sqrt[3]{y}$.

244. But if the radical quantities are connected with others by the signs + and —, they must be involved by a multiplication of the several terms, as in Art. 172.

15. Required the square of $a+\sqrt{y}$ and $a-\sqrt{y}$.

$$\begin{array}{ll}
a+\sqrt{y} & \quad a-\sqrt{y} \\
a+\sqrt{y} & \quad a-\sqrt{y} \\
\hline
a^2+a\sqrt{y} & \quad a^2-a\sqrt{y} \\
\quad a\sqrt{y}+y & \quad -a\sqrt{y}+y \\
\hline
a^2+2a\sqrt{y}+y & \quad a^2-2a\sqrt{y}+y
\end{array}$$

QUEST.—How is a root raised to a power of the same name? If the radicals have co-efficients, how proceed? If the radicals are compound quantities, how?

16. Required the cube of $a-\sqrt{b}$.

17. Required the cube of $2d+\sqrt{x}$.

18. Required the 4th power of $\sqrt{d}$.

19. Required the 4th power of $-\sqrt{ax-1}$.

20. Required the 6th power of $\sqrt{a+b}$.

EVOLUTION OF RADICAL QUANTITIES.

245. The operation for finding the *root* of a quantity which is *already a root*, is the same as in other cases of evolution. Hence we derive the following

RULE FOR THE EVOLUTION OF RADICALS.

I. *Divide the fractional index of the quantity by the number expressing the root to be found.* Or,

Place the radical sign belonging to the required root over the given quantity.

II. If the quantities have rational co-efficients, *the root of these must be extracted, and placed before the radical sign, or quantity.* (Art. 210.)

1. Thus, the square root of $a^{\frac{1}{3}}$, is $a^{\frac{1}{3}\div 2}=a^{\frac{1}{6}}$.

2. Required the cube root of $a(xy)^{\frac{1}{2}}$.

3. Required the nth root of $a\sqrt[6]{by}$.

4. Required the 4th root of $\sqrt{a}\times\sqrt[6]{b}$.

5. Required the 7th root of $128\sqrt[3]{d}$.

245.*a*. From the preceding rules, it will be perceived that *powers* and *roots* may be brought promiscuously together, and subjected to the same modes of operation.

QUEST.—General rule for the evolution of radicals?

EXAMPLES FOR PRACTICE

1. Find the 4th root of $81a^2$.
2. Find the 6th root of $(a+b)^{-3}$.
3. Find the nth root of $(x-y)^{\frac{1}{3}}$.
4. Find the cube root of $-125a^3x^6$.
5. Find the square root of $\frac{4a^4}{9x^2y^2}$.
6. Find the 5th root of $\frac{32a^5x^{10}}{243}$
7. Find the square root of $x^2-6bx+9b^2$.
8. Find the square root of $a^2+ay+\frac{y^2}{4}$.
9. Reduce ax^2 to the form of the 6th root.
10. Reduce $-3y$ to the form of the cube root.
11. Reduce a^2 and $a^{\frac{1}{3}}$ to a common index.
12. Reduce $4^{\frac{1}{3}}$ and $5^{\frac{1}{4}}$ to a common index.
13. Reduce $a^{\frac{1}{2}}$ and $b^{\frac{1}{4}}$ to the common index $\frac{1}{8}$.
14. Reduce $2^{\frac{1}{2}}$ and $4^{\frac{1}{4}}$ to the common index $\frac{1}{8}$.
15. Remove a factor from $\sqrt{294}$.
16. Remove a factor from $\sqrt{x^3-a^2x^2}$.
17. Find the sum and difference of $\sqrt{16a^2x}$ and $\sqrt{4a^2x}$.
18. Find the sum and difference of $\sqrt[3]{192}$ and $\sqrt[3]{24}$.
19. Multiply $7\sqrt[3]{18}$ into $5\sqrt[3]{4}$.
20. Multiply $4+2\sqrt{2}$ into $2-\sqrt{2}$.
21. Multiply $a(a+\sqrt{c})^{\frac{1}{2}}$ into $b(a-\sqrt{c})^{\frac{1}{2}}$.

22. Multiply $2(a+b)^{\frac{1}{n}}$ into $3(a+b)^{\frac{1}{m}}$.
23. Divide $6\sqrt{54}$ by $3\sqrt{2}$.
24. Divide $4\sqrt[3]{72}$ by $2\sqrt[3]{18}$.
25. Divide $\sqrt{7}$ by $\sqrt[3]{7}$.
26. Divide $8\sqrt[3]{512}$ by $4\sqrt[3]{2}$.
27. Find the cube of $17\sqrt{21}$.
28. Find the square of $5+\sqrt{2}$.
29. Find the 4th power of $\frac{1}{6}\sqrt{6}$.
30. Find the cube of $\sqrt{x}-\sqrt{b}$.

SECTION X.

REDUCTION OF EQUATIONS BY INVOLUTION.

ART. 246. In an equation, the letter which expresses the unknown quantity is sometimes found under a *radical sign*. We may have $\sqrt{x}=a$.

To clear this of the radical sign, let each member of the equation be squared, that is, multiplied into itself. We shall then have $\sqrt{x}\times\sqrt{x}=aa$. Or, (Art. 243,) $x=a^2$.

The equality of the sides is not affected by this operation, because each is only multiplied into itself, that is, equal quantities are multiplied into equal quantities. (Ax. 3.)

The same principle is applicable to any root whatever. If $\sqrt[n]{x}=a$; then $x=a^n$. For by Art. 243, a root is raised to a power of the same name, by removing the index or radical sign. Hence,

247. To reduce an equation when the unknown quantity is under a radical sign.

Involve both sides to a power of the same name, as the root expressed by the radical sign.

N. B. It will generally be expedient to make the necessary transpositions, and to clear the equation of fractions, *before* involving the quantities; so that all those which are not under the radical sign may stand on one side of the equation.

1. Reduce the equation $\sqrt{x}+4=9$
   Transposing $+4$ $\sqrt{x}=9-4=5$
   Involving both sides, $x=5^2=25$. Ans.
2. Reduce the equation $a+\sqrt[n]{x}-b=d$
   By transposition $\sqrt[n]{x}=d+b-a$
   By involution $x=(d+b-a)^n$. Ans.
3. Reduce the equation $\sqrt[3]{x+1}=4$.
4. Reduce the equation $4+3\sqrt{x-4}=6+\frac{1}{2}$.
5. Reduce the equation $\sqrt{a^2+\sqrt{x}}=\dfrac{3+d}{\sqrt{(a^2+\sqrt{x})}}$.
6. Reduce $3+2\sqrt{x}-\frac{4}{5}=6$.
7. Reduce $4\sqrt{\dfrac{x}{5}}=8$.
8. Reduce $(2x+3)^{\frac{1}{3}}+4=7$.
9. Reduce $\sqrt{12+x}=2+\sqrt{x}$.
10. Reduce $\sqrt{x-a}=\sqrt{x}-\frac{1}{2}\sqrt{a}$.
11. Reduce $\sqrt{5}\times\sqrt{x+2}=2+\sqrt{5x}$.
12. Reduce $\dfrac{x-ax}{\sqrt{x}}=\dfrac{\sqrt{x}}{x}$.

QUEST.—When the unknown quantity is under the radical sign, how is the equation reduced? What preparation is it advisable to make before involving the quantities?

13. Reduce $\frac{\sqrt{x}+28}{\sqrt{x}+4}=\frac{\sqrt{x}+38}{\sqrt{x}+6}$.

14. Reduce $\sqrt{x}+\sqrt{a+x}=\frac{2a}{\sqrt{(a+x)}}$.

15. Reduce $x+\sqrt{a^2+x^2}=\frac{2a^2}{\sqrt{(a^2+x^2)}}$.

16. Reduce $x+a=\sqrt{a^2+x\sqrt{(b^2+x^2)}}$.

17. Reduce $\sqrt{2+x}+\sqrt{x}=\frac{4}{\sqrt{(2+x)}}$.

18. Reduce $\sqrt{x-32}=16-\sqrt{x}$.

19. Reduce $\sqrt{4x+17}=2\sqrt{x}+1$.

20. Reduce $\frac{\sqrt{(6x)}-2}{\sqrt{(6x)}+2}=\frac{4\sqrt{(6x)}-9}{4\sqrt{(6x)}+6}$.

REDUCTION OF EQUATIONS BY EVOLUTION.

248. In many equations, the letter which expresses the unknown quantity is involved in some power. Thus,

In the equation $x^2=16$,

We have the value of the *square* of x, but not of x itself. If the square root of both sides be extracted,

We shall have $x=4$.

The equality of the members is not affected by this reduction. For if two quantities or sets of quantities are equal, their roots are also equal.

If $(x+a)^n=b+h$, then $x+a=\sqrt[n]{b+h}$. Hence,

249. To reduce an equation when the unknown quantity is a power.

Extract the root of both sides which corresponds with the power expressed by the index of the unknown quantity.

Quest.—When the unknown quantity is a power, how is the equation reduced?

1. Reduce the equation $6+x^2-8=7$

By transposition, $x^2=7+8-6=9$

By evolution, $x=\pm\sqrt{9}=\pm 3$. Ans.

The signs $+$ and $-$ are both placed before $\sqrt{9}$, because an even root of an affirmative quantity is *ambiguous*. (Art. 212.)

2. Reduce the equation $5x^2-30=x^2+34$

Transposing, &c., $x^2=16$

By evolution, $x=\pm 4$. Ans.

3. Reduce the equation $a+\dfrac{x^2}{b}=h-\dfrac{x^2}{d}$.

4. Reduce the equation $a+dx^n=10-x^n$.

250. From the preceding articles it will be easy to see, that to reduce an equation containing a root of a power, (Art. 206), requires both *involution* and *evolution*.

5. Reduce the equation $\sqrt[3]{x^2}=4$

By involution, $x^2=4^3=64$

By evolution, $x=\pm\sqrt{64}=\pm 8$. Ans.

6. Reduce the equation $\sqrt{x^m-a}=h-d$.

7. Reduce the equation $(x+a)^{\frac{1}{2}}=\dfrac{a+b}{(x-a)^{\frac{1}{2}}}$

8. Reduce the equation $(x^2-1)^{\frac{1}{2}}=\dfrac{8}{(x^2-1)^{\frac{1}{2}}}$.

9. Reduce the equation $\sqrt{x^2-11}=5$.

10. Reduce the equation $\sqrt{y^2-4ab}=a-b$.

11. Reduce the equation $(13+\sqrt{23+y^2})^{\frac{1}{2}}=5$.

12. Reduce the equation $(3+\sqrt[3]{329+\sqrt{x}})^2=144$.

QUEST.—Why are the signs $+$ and $-$ placed before the root? How is an expression containing a root of a power reduced?

PROBLEMS.

Prob. 1. A gentleman being asked his age, replied, "If you add to it 10 years, and extract the square root of the sum, and from this root subtract 2, the remainder will be 6." What was his age?

By the conditions of the problem, $\sqrt{x+10}-2=6$

By transposition, $\sqrt{x+10}=6+2=8$

By involution, $x+10=8^2=64$

And $x=64-10=54$

Proof. (Art. 161.) $\sqrt{54+10}-2=6$

Prob. 2. If to a certain number 22577 be added, and the square root of the sum be extracted, and from this 163 be subtracted, the remainder will be 237. What is the number?

Let $x=$ the number sought, $b=163$

$a=22577$, (Art. 159), $c=237$

By the conditions proposed, $\sqrt{x+a}-b=c$

By transposition, $\sqrt{x+a}=c+b$

By involution, $x+a=(c+b)^2$

And $x=(c+b)^2-a$

Restoring the numbers, (Art. 35), $x=(237+163)^2-22577$

That is $x=160000-22577=137423$.

Proof. $\sqrt{137423+22577}-163=237$.

251. When an equation is reduced by extracting an even root of a quantity, the solution does not always determine whether the answer is positive or negative. (Art. 212.) But what is thus left ambiguous by the algebraic process, is frequently settled by the statement of the problem.

Prob. 3. A merchant gains in trade a sum to which 320 dollars bears the same proportion as five times this sum does to 2500. What is the amount gained?

Prob. 4. The distance to a certain place is such, that if 96 be subtracted from the square of the number of miles, the remainder will be 48. What is the distance?

Prob. 5. If three times the square of a certain number be divided by 4, and if the quotient be diminished by 12, the remainder will be 180. What is the number?

Prob. 6. What number is that, the fourth part of whose square being subtracted from 8, leaves a remainder equal to 4?

Prob. 7. What two numbers are those, whose sum is to the greater as 10 to 7; and whose sum multiplied into the less produces 270?

Prob. 8. What two numbers are those, whose difference is to the greater as 2 to 9, and the difference of whose squares is 128?

Prob. 9. It is required to divide the number 18 into two such parts, that the squares of those parts may be to each other as 25 to 16.

Prob. 10. It is required to divide the number 14 into two such parts that the quotient of the greater divided by the less, may be to the quotient of the less divided by the greater as 16 to 9.

Prob. 11. What two numbers are as 5 to 4, the sum of whose cubes is 5103?

Prob. 12. Two travellers, A and B, set out to meet each other, A leaving the town C at the same time that B left D. They travelled the direct road between C and D; and on meeting, it appeared that A had travelled 18 miles more than B, and that A could have gone B's distance in $15\frac{3}{4}$ days,

but B would have been 28 days in going A's distance. Required the distance between C and D. = 126

Prob. 13. Find two numbers which are to each other as 8 to 5, and whose product is 360.

Prob. 14. A gentleman bought two pieces of silk, which together measured 36 yards. Each of them cost as many shillings by the yard as there were yards in the piece, and their whole prices were as 4 to 1. What were the lengths of the pieces?

Prob. 15. Find two numbers which are to each other as 3 to 2; and the difference of whose fourth powers is to the sum of their cubes, as 26 to 7.

Prob. 16. Several gentlemen made an excursion, each taking the same sum of money. Each had as many servants attending him as there were gentlemen; the number of dollars which each had was double the number of all the servants, and the whole sum of money taken out was 3456 dollars. How many gentlemen were there?

Prob. 17. A detachment of soldiers from a regiment being ordered to march on a particular service, each company furnished four times as many men as there were companies in the whole regiment; but these being found insufficient, each company furnished three men more; when their number was found to be increased in the ratio of 17 to 16. How many companies were there in the regiment? = 12

AFFECTED QUADRATIC EQUATIONS.

252. Equations are divided into classes, which are distinguished from each other by the power of the letter that expresses the unknown quantity. Those which contain only

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QUEST.—Into what are equations divided?

the *first* power of the unknown quantity are called *simple* equations, or equations of the *first degree.* Those in which the highest power of the unknown quantity is a *square*, are called *quadratic*, or equations of the *second degree*; those in which the highest power is a *cube* are called *cubic*, or equations of the *third degree*, &c.

Thus $x=a+b$, is an equation of the *first* degree.

$x^2=c$, and $x^2+ax=d$,

are *quadratic* equations, or equations of the *second* degree.

$x^3=h$, and $x^3+ax^2+bx=d$,

are *cubic* equations, or equations of the *third* degree.

253. Equations are also divided into *pure* and *affected* equations. A pure equation contains only *one power* of the unknown quantity. This may be the first, second, third, or any other power. An affected equation contains *different powers* of the unknown quantity. Thus,

$$\left\{\begin{array}{l} x^2=d-b, \text{ is a pure quadratic equation.} \\ x^2+bx=d, \text{ an affected quadratic equation.} \end{array}\right.$$

$$\left\{\begin{array}{l} x^3=b-c, \text{ a pure cubic equation.} \\ x^3+ax^2+bx=h, \text{ an affected cubic equation.} \end{array}\right.$$

In a *pure* equation, all the terms which contain the unknown quantity may be united in one, (Art. 185,) and the equation, however complicated in other respects, may be reduced by the rules which have already been given. But in an *affected* equation, as the unknown quantity is raised to *different powers*, the terms containing these powers cannot be united. (Art. 185 *a*.)

QUEST.—What are those called which contain only the first power of the unknown quantity? When the unknown quantity is a square what? When a cube? How else are equations divided? What is a pure equation? What an affected equation?

254. ***An affected quadratic equation is one which contains the unknown quantity in one term, and the square of that quantity in another term.***

The unknown quantity may be originally in *several* terms of the equation. But all these can be reduced to two, one containing the unknown quantity, and the other its square.

255. It has already been shown that a *pure* quadratic is solved by *extracting the root of both sides of the equation.* An *affected* quadratic may be solved in the same way, if the member which contains the unknown quantity is an *exact square.*

Thus the equation $x^2+2ax+a^2=b+h$, may be reduced by evolution. For the first member is the *square* of a *binomial* quantity. (Art. 173.) And its root is $x+a$. Therefore,

$x+a=\sqrt{b+h}$, and by transposing a,

$x=\sqrt{b+h}-a.$

256. But it is not often the case, that the member of an affected quadratic containing the unknown quantity, is an *exact square*, till an additional term is applied, for the purpose of making the required reduction.

In the equation $x^2+2ax=b$, the side containing the unknown quantity is not a complete square. The two terms of which it is composed, are indeed such as might belong to the square of a binomial quantity. (Art. 173.) But one term is *wanting*. We have then to inquire, in what way this may be supplied. From having *two* terms of the square of a binomial given, how shall we find the *third?*

Of the three terms, two are complete powers, and the other is twice the product of the roots of these powers, or

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QUEST.—What is an affected quadratic equation? How is a pure quadratic equation solved? How an affected quadratic, when it is an exact square? How, when it is not an exact square?

which is the same thing, the product of one of the roots into twice the other.

In the expression x^2+2ax, the term $2ax$ consists of the factors, $2a$ and x. The latter is the unknown quantity. The other factor $2a$ may be considered the *co-efficient* of the unknown quantity ; a co-efficient being another name for a factor. (Art. 24.) As x is the root of the first term x^2 ; the other factor $2a$ is *twice* the root of the third term, which is wanted to complete the square. Therefore *half* of $2a$ is the root of the deficient term, and a^2 is the term itself.

The square completed is $x^2+2ax+a^2$, where it will be seen that the last term a^2 is the square of half of $2a$, and $2a$ is the co-efficient of x, the root of the first term.

In the same manner, it may be proved, that the last term of the square of any binomial quantity, is equal to the square of half the co-efficient of the root of the first term.

257. From this principle is derived the following

METHOD FOR COMPLETING THE SQUARE.

Take the square of half the co-efficient of the first power of the unknown quantity, and add it to both sides of the equation.

258. It will be observed that there is nothing *peculiar* in the solution of *affected quadratics*, except the *completing of the square. Quadratic* equations are *formed* in the same manner as *simple* equations; and after the square is completed, they are reduced in the same manner as pure equations.

1. Reduce the equation $x^2+6ax=b$

Completing the square, $x^2+6ax+9a^2=9a^2+b$

Extracting both sides, (Art. 255,) $x+3a=\pm\sqrt{9a^2+b}$

And $x=-3a\pm\sqrt{9a^2+b}$. Ans.

Here the co-efficient of x, in the given equation, is $6a$.

QUEST.—What is the first method for completing the square? What is there peculiar in the solution of quadratics?

The square of half this, is $9a^2$, which being added to both sides completes the square. The equation is then reduced by extracting the root of each member, in the same manner as in Art. 249, excepting that the square here being that of a *binomial*, its root is found by the rule in Art. 216.

2. Reduce the equation $x^2-8bx=h$.

3. Reduce the equation $x^2+ax=b+h$.

Completing the square, $x^2+ax+\frac{a^2}{4}=\frac{a^2}{4}+b+h$.

4. Reduce the equation $x^2-x=h-d$.

5. Reduce the equation $x^2+3x=d+6$.

6. Reduce the equation $x^2-abx=ab-cd$.

7. Reduce the equation $x^2+\frac{ax}{b}=h$.

8. Reduce the equation $x^2-\frac{x}{b}=7h$.

259. In these and similar instances, the root of the third term of the completed square is easily found, because this root is the same half co-efficient from which the term has just been derived. (Art. 257.) Thus in the last example, half the co-efficient of x is $\frac{1}{2b}$, and this is the root of the third term $\frac{1}{4b^2}$.

260. When the first power of the unknown quantity is in *several terms*, these should be united in one, if they can be by the rules for reduction in addition. But if there are *literal* co-efficients, these may be considered as constituting,

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QUEST.—How do you know what the root of the third term of the completed square is? When the first power is in several terms, what is to be done? If there are literal co-efficients what?

together, a *compound* co-efficient or factor, into which the unknown quantity is multiplied.

Thus $ax+bx+dx=(a+b+d)\times x$. (Art. 97.) The square of half this compound co-efficient is to be added to both sides of the equation.

9. Reduce the equation $x^2+3x+2x+x=d$
 Uniting terms $x^2+6x=d$
 Completing the square $x^2+6x+9=9+d$
 And $x=-3\pm\sqrt{9+d}$. Ans.

10. Reduce the equation $x^2+ax+bx=h$
 By Art. 97, $x^2+(a+b)\times x=h$
 Therefore $x^2+(a+b)\times x+\left(\frac{a+b}{2}\right)^2=\left(\frac{a+b}{2}\right)^2+h.$

11. Reduce the equation $x^2+ax-x=b$.

261. *Before* completing the square, the known and unknown quantities must be brought on opposite sides of the equation by transposition; the square of the unknown quantity must also be *positive*, and it is preferable to make it the *first* or *leading* term.

12. Reduce the equation $a+5x-3b=3x-x^2$
 Transp. and uniting terms $x^2+2x=3b-a$
 Completing the square $x^2+2x+1=1+3b-a$
 And $x=-1\pm\sqrt{1+3b-a}$. Ans.

13. Reduce the equation $\frac{x}{2}=\frac{36}{x+2}-4$.

262. If the *highest power* of the unknown quantity has a *co-efficient*, or *divisor*, *before* completing the square it must be freed from these by multiplication or division. (Arts. 149, 154.)

QUEST.—Before completing the square what preparations is it expedient to make? If the highest power has a co-efficient or divisor, what should be done?

14. Reduce the equation $x^2+24a-6h=12x-5x^2$

Transp. and uniting terms $6x^2-12x=6h-24a$

Dividing by 6, $x^2-2x=h-4a$

Completing the square, $x^2-2x+1=1+h-4a$

Extracting and transp. $x=1\pm\sqrt{1+h-4a}$. Ans.

15. Reduce the equation $h+2x=d-\frac{bx^2}{a}$.

273. If the square of the unknown quantity is in *several terms*, the equation must be divided by *all* the co-efficients of this square. (Art. 155.)

16. Reduce the equation $bx^2+dx^2-4x=b-h$

Dividing by $b+d$, $x^2-\frac{4x}{b+d}=\frac{b-h}{b+d}$.

17. Reduce the equation $ax^2+x=h+3x-x^2$.

Given $ax^2+bx=d$, to find x.

If this equation is multiplied by $4a$, and if b^2 is added to both sides, it will become,

$$4a^2x^2+4abx+b^2=4ad+b^2;$$

the first number of which is a complete square of the binomial $2ax+b$.

264. From the foregoing principle is deduced

A SECOND METHOD OF COMPLETING THE SQUARE.

Multiply the equation by 4 times the co-efficient of the highest power of the unknown quantity, and add to both sides the square of the co-efficient of the lowest power.

The advantage of this method is, that it avoids the introduction of *fractions*, in completing the square.

QUEST.—If the square of the unknown quantity is in several terms, how proceed? What is the second method of completing the square? What advantage has this method?

DEMONSTRATION.

1. The object of multiplying the equation by the *co-efficient* of the *highest power*, is to render the first term a perfect square without removing its co-efficient, and at the same time to obtain the middle term of the square of a binomial. But we must multiply all the terms of the equation by this quantity to preserve the equality of its members. (Ax. 3.) The equation above when mult. by a, becomes $a^2x^2+abx=ad$.

That the first term will, in all cases, be rendered a complete square when multiplied by its co-efficient, is evident from the fact, that it will then consist of two factors, each of which is a square, viz. x^2, and the square of its co-efficient. But the product of the squares of two or more factors, is equal to the square of their product. (Art. 167.)

2. It will be seen that one term is still wanting in the first member, in order to make it the square of a binomial, viz. the square of the last term. (Art. 173.)

This deficiency may be supplied by adding to both sides the square of half the co-efficient of the lowest power, as in the first method of completing the square. But in taking half of this co-efficient, the learner will often be encumbered with fractions which it is desirable to avoid. Thus in the equation above, half of the co-efficient of the lowest power is $\frac{b}{2}$, the square of which is $\frac{b^2}{4}$. Adding this to both sides, the equation will become, $a^2x^2+abx+\frac{b^2}{4}=ad+\frac{b^2}{4}$, the first member of which is a complete square of the binomial, $ax+\frac{b}{2}$.

QUEST.—Why is the equation multiplied by the co-efficient of the highest power? How does it appear that this will make the first term an exact square? Why add the square of the co-efficient of the lowest power to both sides?

Now it is obvious, that multiplying the equation by 4, is the same as removing the denominator 4 from the third term. Hence multiplying the equation by 4, will avoid the introduction of fractions, and also leave the square of the whole of the co-efficient of the lowest power to be added to both sides according to the rule.

The first term evidently continues to be a square after it is multiplied by 4, for it is still the product of the *powers* of certain factors. (Art. 167.)

3. It will be perceived at once, that the second term is composed of twice the root of the first term multiplied into the co-efficient of the last term, which constitutes the middle term of a binomial square. (Art. 173.)

Obser. It is manifest from the preceding demonstration, that multiplying by 4 is not a *necessary* step in completing the square, but is resorted to as an expedient to prevent the occurrence of fractions. When therefore the co-efficient of the lowest power is an *even* number, so that half of it can be taken without a remainder, we may *simplify* the operation by multiplying by the co-efficient of the highest power alone, and adding to both sides the square of half the co-efficient of the lowest power of the unknown quantity.

Take the equation $7x^2+40x=71\frac{3}{7}$.

Multiplying by 7, it becomes $\quad 49x^2+280x=500$

Adding the square of half the co-efficient, $\quad 49x^2+280x+400=900$

By evolution and transposition, $\quad x=40.$

265. From the preceding principles we may also deduce

OTHER METHODS OF COMPLETING THE SQUARE.

Multiply the equation by 16 *times the co-efficient of the highest power of the unknown quantity, and add to both sides* 4 *times the square of the co-efficient of the lowest power.*

QUEST.—Why multiply the equation by 4? How may the equation be simplified when the co-efficient of the lowest power is an even number?

And universally, multiplying the equation by the product of any square number, as n^2, into the co-efficient of the highest power, and adding to both sides the square of half the root of this number into the square of the co-efficient of the lowest power, will render it a complete square.

Take the equation $x^2-3x=4$

Multiplying by 16, &c. $16x^2-48x+36=100$

By evolution and transposition, $x=4$

Or, take the equation $ax^2+cx=d$.

Mult. by n^2, &c. $n^2a^2x^2+n^2acx+\frac{n^2c^2}{4}=n^2ad+\frac{n^2c^2}{4}$; the first member of which is the square of the binomial, $nax+\frac{nc}{2}$.

There is an *obvious advantage*, however, in employing 4 in preference to any other square number. For multiplying the equation by 4 times the co-efficient of the highest power, will produce the *middle* term of a binomial square, the *third* term of which is the square of the co-efficient of the lowest power.

18. Reduce the equation $ax^2+dx=h$.
19. Reduce the equation $3x^2+5x=42$.
20. Reduce the equation $x^2-15x=-54$.

265.*a*. In the square of a binomial, the first and last terms are always *positive*. For each is the square of one of the terms of the root, and all even powers are positive. (Arts. 168, 173.)

If then $-x^2$ *occurs in an equation, it cannot with this sign form a part of the square of a binomial.* But if *all* the signs in the equation be changed, whilst the equality of the sides will be preserved, the term $-x^2$ will become positive, and the square may then be completed. (Art. 146.)

QUEST.—What other ways of completing the square are mentioned? In the square of a binomial, what sign have the first and last terms? If the square of the unknown quantity has the sign — before it, what must be done?

21. Reduce the equation $-x^2+2x=d-h$

Changing all the signs $x^2-2x=h-d$.

22. Reduce the equation $4x-x^2=-12$.

266. In a quadratic equation, the first term x^2 is the square of a single letter. But a binomial quantity may consist of terms, one or both of which are already powers.

Thus x^3+a is a binomial, and its square is $x^6+2ax^3+a^2$, where the index of x in the first term is twice as great as in the second. When the third term is deficient, the square may be completed in the same manner as that of any other binomial. For the middle term is twice the product of the roots of the two others.

So the square of x^n+a, is $x^{2n}+2ax^n+a^2$.

And the square of $x^{\frac{1}{n}}+a$, is $x^{\frac{2}{n}}+2ax^{\frac{1}{n}}+a^2$. Therefore,

267. *Any equation which contains only two different powers or roots of the unknown quantity, the index of one of which is twice that of the other, may be solved in the same manner as a quadratic equation, by completing the square.*

N. B. It must be observed, that in the binomial root, the letter expressing the unknown quantity may still have a fractional or integral index, so that a farther operation may be necessary. (Art. 250.)

23. Reduce the equation $x^4-x^2=b-a$.

Completing the square $x^4-x^2+\frac{1}{4}=\frac{1}{4}+b-a$

Extracting and transposing, $x^2=\frac{1}{2}\pm\sqrt{\frac{1}{4}+b-a}$.

Extracting again, (Art. 249,) $x=\pm\sqrt{\frac{1}{2}\pm\sqrt{(\frac{1}{4}+b-a)}}$.

24. Reduce the equation $x^{2n}-4bx^n=a$.

QUEST.—How solve equations which contain only two different powers or roots of the unknown quantity, when the index of one is twice that of the other?

25. Reduce the equation $x+4\sqrt{x}=h-n$.

26. Reduce the equation $x^{\frac{2}{n}}+8x^{\frac{1}{n}}=a+b$.

268. The solution of a quadratic equation, whether pure or affected, gives two results. For after the equation is reduced, it contains an ambiguous root. In a *pure* quadratic, this root is the *whole* value of the unknown quantity.

Thus the equation $x^2=64$,

Becomes when reduced, $x=\pm\sqrt{64}$. (Art. 249.)

That is, the value of x is either $+8$ or -8, for each of these is a root of 64. Here both the values of x are the same, except that they have contrary signs. This will be the case in every pure quadratic equation, because the whole of the second member is under the radical sign. The two values of the unknown quantity will be alike, except that one will be positive, and the other negative.

269. But in *affected* quadratics, a *part* only of one side of the reduced equation is under the radical sign. When this part is added to, or subtracted from, that which is without the radical sign; the two results will differ in quantity, and will have their signs in some cases alike, and in others unlike.

27. The equation $x^2+8x=20$

Becomes when reduced, $x=-4\pm\sqrt{16+20}$.

That is, $x=-4\pm6$.

Here the first value of x is $-4+6=+2$ } one positive, and
And the second is $-4-6=-10$ } the other negative.

28. The equation $x^2-8x=-15$

Becomes when reduced, $x=4\pm\sqrt{16-15}$

That is, $x=4\pm1$.

QUEST.—How many results does the solution of a quadratic give? In pure quadratics, is the whole value ambiguous? Is this the case in affected quadratics?

Here the first value of x is $4+1=+5$ } both positive.
And the second is $4-1=+3$ }

That these two values of x are correctly found, may be proved by substituting first one and then the other, for x itself, in the original equation. (Art. 161.)

$$\text{Thus } 5^2-8\times5=25-40=-15$$
$$\text{And } 3^2-8\times3=9-24=-15.$$

270. In the reduction of an affected quadratic equation, the value of the unknown quantity is frequently found to be *imaginary*.

29. Thus the equation $x^2-8x=-20$.

Becomes when reduced, $x=4\pm\sqrt{16-20}$

That is, $x=4\pm\sqrt{-4}$.

Here the root of the negative quantity -4 cannot be assigned, (Art. 214,) and therefore the value of x cannot be found. There will be the same impossibility, in every instance in which the negative part of the quantities under the radical sign, is greater than the positive part.

271. When *one* of the values of the unknown quantity in a quadratic equation is imaginary, the *other* is so also. For both are equally affected by the imaginary root.

Thus in the example above,

The first value of x is $4+\sqrt{-4}$,

And the second is $4-\sqrt{-4}$; each of which contains the imaginary quantity $\sqrt{-4}$.

272. An equation which when reduced contains an imaginary root, is often of use to enable us to determine whether

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QUEST.—Is the value of the unknown quantity ever imaginary? When one of the values is imaginary, what is true of the other? Are equations containing an imaginary root of any use? What use?

a proposed question admits of an answer, or involves an absurdity.

30. Suppose it is required to divide 8 into two such parts that the product will be 20.

If x is one of the parts, the other will be $8-x$.

By the conditions proposed, $(8-x)\times x=20$

This becomes when reduced, $x=5\pm\sqrt{-4}$.

Here the imaginary expression $\sqrt{-4}$ shows that an answer is impossible; and that there is an absurdity in supposing that 8 may be divided into two such parts that their product shall be 20.

273. Although a quadratic equation gives two results, yet both these may not always be applicable to the subject proposed. The quantity under the radical sign may be produced either from a positive or a negative root. But both these roots may not, in every instance, belong to the problem to be solved. (Art. 251.)

31. Divide the number 30 into two such parts, that their product may be equal to 8 times their difference.

If $x=$ the less, then $30-x=$ the greater part.

By the supposition, $x\times(30-x)=8\times(30-2x)$.

This reduced, gives $x=23\pm17=40$, or 6, the less part.

But as 40 cannot be a part of 30, the problem can have but one real solution, making the less part 6, and the greater part 24.

274. The preceding principles in quadratic equations may be summed up in the following

GENERAL RULE.

1. *Transpose all the unknown quantities to one side of the equation; and the known quantities to the other.*

QUEST.—Are both of the results of a quadratic always applicable to the problem under consideration? What is the general rule for the solution of quadratic equations?

II. *Make the square of the unknown quantity positive* (if it is not already) *by changing the signs of all the terms on both sides; and place it for the first or leading term.* (Art. 261.)

III. To complete the square,

1. *Remove the co-efficient of the second power of the unknown quantity, and add the square of half of the co-efficient of the first power of the unknown quantity to both sides of the equation.* (Art. 257.) Or,

2. *Multiply the equation by four times the co-efficient of the highest power of the unknown quantity, and add to both sides the square of the co-efficient of the first power of the unknown quantity.* (Art. 264.)

IV. *Reduce the equation by extracting the square root of both sides; and transpose the known part of the binomial root thus obtained to the opposite side.* (Art. 255.)

EXAMPLES FOR PRACTICE.

1. Reduce $3x^2-9x-4=80$.

2. Reduce $4x-\frac{36-x}{x}=46$.

3. Reduce $4x-\frac{14-x}{x+1}=14$.

4. Reduce $5x-\frac{3x-3}{x-3}=2x+\frac{3x-6}{2}$.

5. Reduce $\frac{16}{x}-\frac{100-9x}{4x^2}=3$.

6. Reduce $\frac{3x-4}{x-4}+1=10-\frac{x-2}{2}$.

7. Reduce $\frac{x+4}{3}-\frac{7-x}{x-3}=\frac{4x+7}{9}-1$.

8. Reduce $\frac{x^3-10x^2+1}{x^2-6x+9}=x-3$.

9. Reduce $\frac{6}{x+1}+\frac{2}{x}=3.$

10. Reduce $\frac{3x}{x+2}-\frac{x-1}{6}=x-9$

11. Reduce $\frac{x}{a}+\frac{a}{x}=\frac{2}{a}.$

12. Reduce $x^4+ax^2=b.$

13. Reduce $\frac{x^6}{2}-\frac{x^3}{4}=-\frac{1}{32}.$

14. Reduce $2x^{\frac{2}{3}}+3x^{\frac{1}{3}}=2.$

15. Reduce $\frac{1}{2}x-\frac{1}{3}\sqrt{x}=22\frac{1}{6}.$

16. Reduce $2x^4-x^2+96=99.$

17. Reduce $(10+x)^{\frac{1}{2}}-(10+x)^{\frac{1}{4}}=2.$

18. Reduce $3x^{2n}-2x^n=8.$

19. Reduce $2(1+x-x^2)-\sqrt{1+x-x^2}=-\frac{1}{9}.$

20. Reduce $\sqrt[3]{x^3-a^3}=x-b.$

21. Reduce $\frac{\sqrt{4x}+2}{4+\sqrt{x}}=\frac{4-\sqrt{x}}{\sqrt{x}}.$

22. Reduce $x^{\frac{6}{5}}+x^{\frac{3}{5}}=756.$

23. Reduce $\sqrt{2x+1}+2\sqrt{x}=\frac{21}{\sqrt{(2x+1)}}$

24. Reduce $2\sqrt{x-a}+3\sqrt{2x}=\frac{7a+5x}{\sqrt{(x-a)}}.$

25. Reduce $x+16-7\sqrt{x+16}=10-4\sqrt{x+16}.$

26. Reduce $\sqrt{x^5}+\sqrt{x^3}=6\sqrt{x}.$

27. Reduce $\frac{4x-5}{x}-\frac{3x-7}{3x+7}=\frac{9x+23}{13x}.$

28. Reduce $\frac{3}{6x-x^2}+\frac{6}{x^2+2x}=\frac{11}{5x}$.

29. Reduce $(x-5)^3-3(x-5)^{\frac{3}{2}}=40$.

30. Reduce $x+\sqrt{x+6}=2+3\sqrt{x+6}$.

PROBLEMS IN QUADRATIC EQUATIONS.

Prob. 1. A merchant has a piece of cotton cloth, and a piece of silk. The number of yards in both is 110: and if the square of the number of yards of silk be subtracted from 80 times the number of yards of cotton, the difference will be 400. How many yards are there in each piece?

Let $x=$ the yards of silk.

Then $110-x=$ the yards of cotton.

By the supposition $80\times(110-x)-x^2=400$.

Therefore $x=-40\pm\sqrt{10000}=-40\pm 100$.

The first value of x, is $-40+100=60$, the yards of silk;

And $110-x=110-60=50$, the yards of cotton.

The second value of x, is $-40-100=-140$; but as this is a negative quantity, it is not applicable to goods which a man has in his possession.

Prob. 2. The ages of two brothers are such, that their sum is 45 years, and their product 500. What is the age of each?

Prob. 3. To find two numbers such, that their difference shall be 4, and their product 117.

Prob. 4. A merchant having sold a piece of cloth which cost him 30 dollars, found that if the price for which he sold it were multiplied by his *gain*, the product would be equal to the cube of his gain. What was his gain?

Prob. 5. To find two numbers whose difference shall be 3, and the difference of their cubes 117.

Prob. 6. To find two numbers whose difference shall be 12, and the sum of their squares 1424.

Prob. 7. Two persons draw prizes in a lottery, the difference of which is 120 dollars, and the greater is to the less, as the less to 10. What are the prizes?

Prob. 8. What two numbers are those whose sum is 6, and the sum of their cubes 72?

Prob. 9. Divide the number 56 into two such parts, that their product shall be 640.

Prob. 10. A gentleman bought a number of pieces of cloth for 675 dollars, which he sold again at 48 dollars by the piece, and gained by the bargain as much as one piece cost him. What was the number of pieces?

Prob. 11. A and B started together, for a place 150 miles distant. A's hourly progress was 3 miles more than B's, and he arrived at his journey's end 8 hours and 20 minutes before B. What was the hourly progress of each?

Prob. 12. The difference of two numbers is 6; and if 47 be added to twice the square of the less, it will be equal to the square of the greater. What are the numbers?

Prob. 13. A and B distributed 1200 dollars each, among a certain number of persons. A relieved 40 persons more than B, and B gave to each individual 5 dollars more than A. How many were relieved by A and B?

Prob. 14. Find two numbers whose sum is 10, and the sum of their squares 58.

Prob. 15. Several gentlemen made a purchase in company for 175 dollars. Two of them having withdrawn, the bill was paid by the others, each furnishing 10 dollars more than would have been his equal share if the bill had been paid by the whole company. What was the number in the company at first?

Prob. 16. A merchant bought several yards of linen for 60 dollars, out of which he reserved 15 yards, and sold the remainder for 54 dollars, gaining 10 cents a yard. How many yards did he buy, and at what price?

Prob. 17. A and B set out from two towns, which were 247 miles distant, and travelled the direct road till they met. A went 9 miles a day; and the number of days which they travelled before meeting was greater by 3 than the number of miles which B went in a day. How many miles did each travel?

Prob. 18. A gentleman bought two pieces of cloth, the finer of which cost 4 shillings a yard more than the other. The finer piece cost £18; but the coarser one, which was 2 yards longer than the finer, cost only £16. How many yards were there in each piece; and what was the price of a yard of each?

Prob. 19. A merchant bought 54 gallons of Madeira wine, and a certain quantity of Teneriffe. For the former he gave half as many shillings by the gallon, as there were gallons of Teneriffe, and for the latter 4 shillings less by the gallon. He sold the mixture at 10 shillings by the gallon, and lost £28 16*s*. by his bargain. Required the price of the Madeira, and the number of gallons of Teneriffe.

Prob. 20. If the square of a certain number be taken from 40, and the square root of this difference be increased by 10, and the sum be multiplied by 2, and the product divided by the number itself, the quotient will be 4. What is the number?

Prob. 21. A person being asked his age, replied, If you add the square root of it to half of it, and subtract 12, the remainder will be nothing. What was his age?

Prob. 22. Two casks of wine were purchased for 58 dollars, one of which contained 5 gallons more than the other, and the price by the gallon was 2 dollars less than $\frac{1}{3}$ of the

number of gallons in the smaller cask. Required the number of gallons in each, and the price by the gallon.

Prob. 23. In a parcel which contains 24 coins of silver and copper, each silver coin is worth as many cents as there are copper coins, and each copper coin is worth as many cents as there are silver coins; and the whole is worth 2 dollars and 16 cents. How many are there of each ?

Prob. 24. A person bought a certain number of oxen for 80 guineas. If he had received 4 more oxen for the same money, he would have paid one guinea less for each. What was the number of oxen ?

Prob. 25. It is required to divide 24 into two such parts that their product shall be equal to 35 times their difference.

Prob. 26. The sum of two numbers is 60, and their product is to the sum of their squares as 2 to 5. What are the numbers ?

Prob. 27. Divide 146 into two such parts, that the difference of their square roots may be 6. 121 25

Prob. 28. What two numbers are those whose difference is 16, and their product 36 ?

Prob. 29. Find two numbers whose sum shall be $1\frac{1}{3}$ and the sum of their reciprocals $3\frac{1}{5}$.

Prob. 30. Required to find two numbers whose difference is 15, and half of their product is equal to the cube of the less number ?

Prob. 31. A company incurred a bill of £8 8s. One of them absconded before it was paid, and in consequence, those who remained had to pay 4s. a piece more than their just share. How many were there in the company ?

Prob. 32. A gentleman bequeathed £7 4s. to his grandchildren; but before the money was distributed two more were

added to their number, and consequently the former received one shilling a piece less than they otherwise would have done. How many grandchildren did he leave?

Prob. 33. The length added to the breadth of a rectangular room makes 42 feet, and the room contains 432 square feet. Required the length and breadth.

Prob. 34. A says to B, "the product of our years is 120; and if I were 3 years younger, and you were 2 years older, the product of our ages would still be 120." How old was each?

Prob. 35. Should the square of a certain number be taken from 89, and the square root of their difference be increased by 12, and the sum multiplied by 4, and the product divided by the number itself, the quotient will be 16. What is the number?

Prob. 36. A mason laid 105 rods of wall, and on reflection found that if he had laid 2 rods less per day, he would have been 6 days longer in accomplishing the job. How many rods did he build per day?

Prob. 37. The length of a gentleman's garden exceeded its breadth by 5 rods. It cost him 3 dols. per rod to fence it; and the whole number of dollars which the fence cost, was equal to the number of square rods in the garden. What were its length and breadth?

Prob. 38. What number is that, which being added to its square root will make 156?

Prob. 39. The circumference of a grass-plot is 48 yards, and its area is equal to 35 times the difference of its length and breadth. What are its length and breadth?

Prob. 40. A gentleman purchased a building lot, and in the centre of it, erected a house 54 feet long and 36 feet wide,

which covered just one half his land. This arrangement left him a flower border of uniform width all round his house. What was the width of his border, what the length and breadth of his lot, and how much land did he buy?

Prob. 41. A general wished to arrange his army, which consisted of ~~1200~~ men, in a solid body, so that each rank should exceed each file by 59 men. How many must he place in rank and file?

Prob. 42. A man has a painting 18 inches long, and 12 inches wide, which he orders the cabinet-maker to put into a frame of uniform width, and to have the area of the frame equal to that of the painting. Of what width will the frame be? 3.

Prob. 43. A and B together invest $500 in business, of which each put in a certain share. A's money continued in trade 5 months, B's only two months, and each received back $450 for his capital and profit. What share of the stock did each contribute?

Prob. 44. A merchant sold a quantity of goods for £39, and gained as much per cent. as the goods cost him. How much did he pay for the goods?

Prob. 45. A farmer bought a flock of sheep for £60. After selecting 15 of the best, he sold the remainder for £54, and gained thereby 2 shillings a head. How many sheep did he buy, and what was the price of each?

Prob. 46. A and B started from two cities 247 miles apart, and travelled the same road till they met. A's progress was 1 m. per day less than B's, and the number of days before they met was greater by 3 than the number of miles B went per day. How many miles did each travel?

Prob. 47. Two persons, A and B, invest $900 in business. A's money remained in trade 4 months, and he received $512

for his share of the profit and stock; B's money was in trade 7 months, and he received $469 for his share of the profit and stock. What was each partner's stock?

Prob. 48. A merchant bought a piece of cloth for $54; the number of shillings which he paid per yard was $\frac{4}{9}$ of the number of yards. Required the length of the cloth, and the price per yard.

Prob. 49. There was a cask containing 20 gallons of wine? a quantity of this was drawn off and put into another cask of equal size, and then this last was filled with water; and afterwards the first cask was filled with the mixture from the second. It appears that if $6\frac{2}{3}$ gallons are now drawn from the first and put into the second, there will be equal quantities of wine in each cask. How much wine was first drawn off? 10

Prob. 50. A man bought 80 lbs. of pepper and 100 lbs. of ginger for $65, at such prices that he obtained 60 lbs. more of ginger for £20 than he did of pepper for £10. What did he pay per pound for each?

SECTION XI.

TWO UNKNOWN QUANTITIES.

275. In the examples given in the preceding sections, each problem has contained only *one* unknown quantity. Or if, in some instances, there have been *two*, they have been so related to each other, that they have both been expressed by means of the same letter.

But cases frequently occur, in which *two* unknown quantities are necessarily introduced into the same calculation.

Suppose the following equations are given.

1. $x+y=14$
2. $x-y=2$.

If y be transposed in each, they will become

1. $x=14-y$
2. $x=2+y$.

Here the first member of each of the equations is x, and the second member of each is *equal* to x. But according to Axiom 7, quantities which are respectively equal to any other quantity, are equal to each other; therefore,

$$2+y=14-y, \text{ and } y=6.$$

By substituting the value of y in the 1st equation, (Art. 159,) we have $x+6=14$; then $x=8$.

276. In solving the preceding problem, it will be observed that we first found the value of the unknown quantity x, in each equation; and then by making one of the expressions denoting the value of x, equal to the other, (Axiom 7,) we formed a new equation, which contained only the other unknown quantity y.

This process is called *extermination* or *elimination.*

There are three methods of extermination, viz. by comparison, by substitution, and by addition and subtraction.

EXTERMINATION BY COMPARISON.

277. Case I. To exterminate one or two unknown quantities by comparison.

Quest.—How are problems solved which contain two unknown quantities? What is this process called? How many methods of extermination? Name them.

15*

Find the value of one of the unknown quantities in each of the equations, and form a new equation by making one of these values equal to the other.

Prob. 1. Given $\left.\begin{array}{l} x+y=36 \\ x-y=12 \end{array}\right\}$ To find the value of x and y.

1. In the first equation, $x+y=36$
2. In the second equation, $x-y=12$
3. Transposing y in first equation, $x=36-y$
4. Transposing y in second equation, $x=12+y$
5. Making 3d and 4th equal, (Ax. 7,) $12+y=36-y$
6. Transposing, &c., $y=12$

Substituting the value of y in the 4th, $x=12+12=24$.

Prob. 2. Given $\left.\begin{array}{l} 2x+3y=28 \\ 3x+2y=27 \end{array}\right\}$ To find the value of x and y.

Prob. 3. Given $\left.\begin{array}{l} 4x+y=43 \\ 5x+2y=56 \end{array}\right\}$ To find the value of x and y.

Prob. 4. Given $\left.\begin{array}{l} 4x-2y=16 \\ 6x=9y \end{array}\right\}$ To find the value of x and y.

Prob. 5. Given $\left.\begin{array}{l} 4x-2y=20 \\ 4x+2y=100 \end{array}\right\}$ To find the value of x and y.

Prob. 6. Given $\left.\begin{array}{l} 5x+8=7y \\ 5y+32=7x \end{array}\right\}$ To find the value of x and y.

Prob. 7. To find two numbers such, that their sum shall be 24; and the greater shall be equal to five times the less.

Let $x=$ the greater; and $y=$ the less.

Prob. 8. To find one of two quantities, whose sum is equal to h; and the difference of whose squares is equal to d. $x=\frac{h^2+d}{2h}$ $y=\frac{h^2-d}{2h}$

Prob 9. Given $\left.\begin{array}{l} ax+by=h \\ x+y=d \end{array}\right\}$ To find y. $y=\frac{h-ad}{b-a}$ $y=\frac{db-h}{b-a}$

QUEST.—What is the rule to exterminate one of two unknown quantities by comparison?

278. When the value of one of the unknown quantities is determined, the other may be easily obtained by substituting, in one of the previous equations, the *value* of the one found for the *quantity itself.* (Art. 159.)

The rule given above, may be generally applied for the extermination of unknown quantities. But there are cases in which other methods will be found more expeditious.

Prob. 10. Suppose $x=hy$
And $ax+bx=y^2$

As in the first of these equations x is equal to hy, we may in the second equation *substitute* this value of x for x itself. The second equation will then become, $ahy+bhy=y^2$.

The equality of the two sides is not affected by this alteration, because we only change one quantity x for another which is equal to it. By this means we obtain an equation which contains only one unknown quantity.

This process is called *extermination by substitution.* Hence,

279. Case II. To exterminate an unknown quantity by substitution.

Find the value of one of the unknown quantities, in one of the equations; and then in the other equation, SUBSTITUTE *this value for the unknown quantity itself.* (Art. 159.)

Prob. 11. Given $x+3y=15$ And $4x+5y=32$ } To find the value of x and y.

Transposing $3y$ in the 1st equation, $x=15-3y$.

Substituting the value of x in the 2d equation, (Art. 159,) we have $60-12y+5y=32$

Then $y=4$

And $x=15-12=3$.

Quest.—After the value of one unknown quantity is found, how obtain the other? What is the second method of extermination called? What is the rule?

Prob. 12. Given $8x+y=42$ And $2x+4y=18$ } To find the value of x and y.

Prob. 13. Given $2x+8y=84$ And $4x+6y=68$ } To find the val. of x and y.

Prob. 14. Given $3x+3y=72$ And $4x+5y=116$ } To find the val. of x and y.

Prob. 15. Given $\frac{1}{2}x+10y=124$ And $2x+9y=124$ } To find the val. of x and y.

Prob. 16. A privateer in chase of a ship 20 miles distant, sails 8 miles, while the ship sails 7. How far will each sail before the privateer will overtake the ship?

Prob. 17. The ages of two persons, A and B, are such that seven years ago, A was three times as old as B; and seven years hence, A will be twice as old as B. What is the age of each? A = 49 B = 21

Prob. 18. There are two numbers, of which the greater is to the less as 3 to 2; and their sum is the 6th part of their product. What are the numbers? x = 15 y = 10

280. There is a *third* method of exterminating an unknown quantity from an equation, which, in many cases, is preferable to either of the preceding.

Prob. 19. Suppose that $x+3y=a$
And $x-3y=b$

If we *add together* the first members of these two equations, and also the second members, we shall have

$$2x=a+b,$$

an equation which contains only the unknown quantity x. The other, having equal co-efficients with contrary signs, has disappeared. (Art. 54.) The equality of the sides is preserved because we have only added equal quantities to equal quantities.

Again, suppose $3x+y=h$

And $2x+y=d$

If we *subtract* the last equation from the first, we shall have

$$x=h-d,$$

where y is exterminated, without affecting the equality of the sides.

Again, suppose $x-2y=a$

And $x+4y=b$

Multiplying the 1st by 2, $2x-4y=2a$,

Then adding the 2d and 3d, $3x=b+2a$.

This process is called *extermination by addition and subtraction*. Hence,

281. CASE III. To exterminate an unknown quantity by addition and subtraction.

Multiply or divide the equations, if necessary, in such a manner that the term which contains one of the unknown quantities, shall be the same in both equations.

Then subtract one equation from the other, if the signs of this unknown quantity are alike, or add them together, if the signs are unlike.

N. B. It must be kept in mind that both members of an equation are always to be increased or diminished alike, in order to preserve their equality.

Prob. 20. Given $\left.\begin{array}{l} 2x+4y=20 \\ \text{And} \quad 4x+5y=28 \end{array}\right\}$ To find the value of x and y.

1. Mult. the 1st equation by 2, $4x+8y=40$

2. The 2d equation is $4x+5y=28$

Subtracting the 2d from the 1st, $3y=12$

Dividing, &c. $y=4$; and $x=2$.

QUEST.—What is the third method of extermination called? What is the rule? What is the object of multiplying the equation by a certain quantity? How do you know when to add and when to subtract?

Prob. 21. Given $\left.\begin{array}{l}2x+y=16 \\ \text{And } 3x-3y=6\end{array}\right\}$ To find the value of x and y.

Prob. 22. Given $\left.\begin{array}{l}4x+3y=50 \\ \text{And } 3x-3y=6\end{array}\right\}$ To find the value of x and y.

Prob. 23. Given $\left.\begin{array}{l}3x+2y=38 \\ \text{And } 5x+4y=68\end{array}\right\}$ To find the value of x and y.

Prob. 24. Given $\left.\begin{array}{l}4x-40=-4y \\ \text{And } 6x-63=-7y\end{array}\right\}$ To find the val. of x and y.

Prob. 25. The numbers of two opposing armies are such, that the sum of both is 21110; and twice the number in the greater army, added to three times the number in the less, is 52219. What is the number in each army?

Prob. 26. A boy purchased 8 lemons and 4 oranges for 56 cents. He afterwards bought 3 lemons and 8 oranges for 60 cents. What did he pay for each?

Prob. 27. The sum of two numbers is 220, and if 3 times the less be taken from 4 times the greater, the remainder will be 180. What are the numbers?

282. In the solution of the succeeding problems, either of the three rules for exterminating unknown quantities may be used at pleasure.

N. B. *That quantity which is the least involved should be the one chosen to be exterminated first.*

The pupil will find it a useful exercise to solve each example by each of the several methods, and carefully observe which is the most comprehensive, and the best adapted to different classes of problems.

Prob. 28. The mast of a ship consists of two parts: one-third of the lower part added to one-sixth of the upper part, is equal to 28 feet; and five times the lower part, diminished by six times the upper part, is equal to 12 feet. What is the height of the mast?

Prob. 29. To find a fraction such that, if a unit be added to the numerator, the fraction will be equal to $\frac{1}{3}$; but if a unit be added to the denominator, the fraction will be equal to $\frac{1}{4}$.

Let $x=$ the numerator, And $y=$ the denominator.

1. By the first condition, $\frac{x+1}{y}=\frac{1}{3}$

By the second, $\frac{x}{y+1}=\frac{1}{4}$

3. Therefore $x=4$, the numerator. } Ans.
4. And $y=15$, the denominator. }

Prob. 30. What two numbers are those, whose *difference* is to their sum as 2 to 3; whose sum is to their product as 3 to 5?

Prob. 31. To find two numbers such, that the product of their sum and difference shall be 5, and the product of the sum of their squares and the difference of their squares shall be 65. $x=3$. $y=2$

Prob. 32. To find two numbers whose sum is 32, and whose product is 240.

Prob. 33. To find two numbers whose sum is 52, and the sum of their squares 1424.

Prob. 34. A certain number consists of two digits or figures, the sum of which is 8. If 36 be added to the number, the digits will be inverted. What is the number?

Prob. 35. The united ages of A and B amount to a certain number of years consisting of two digits, the sum of which is 9. If 27 years be subtracted from the amount of their ages, the digits will be inverted. What is the sum of their ages?

Prob. 36. A merchant having mixed a quantity of brandy and gin, found if he had put in 6 gallons more of each, the compound would have contained 7 gallons of brandy for

every 6 of gin; but if he had put in 6 gallons less of each, the proportions would have been as 6 to 5. How many gallons did he mix of each? $x=76$ $y=66$

THREE UNKNOWN QUANTITIES.

283. In the preceding examples of two unknown quantities, it will be perceived that the conditions of each problem have furnished two equations independent of each other. It often becomes necessary to introduce *three* or *more* unknown quantities into a calculation. In such cases, if the problem admits of a determinate answer, there will always arise from the conditions as many equations *independent* of each other, as there are unknown quantities.

284. Equations are said to be *independent* when they express *different* conditions.

They are said to be *dependent* when they express the *same* conditions under *different* forms. The former are not convertible into each other; but the latter may be changed from one form to the other. Thus $b-x=y$; and $b=y+x$, are dependent equations, because one is formed from the other by merely transposing x.

Obser. Equations are said to be identical when they express the same thing in the same form; as $4x-6=4x-6$.

Prob. 37. Suppose $x+y+z=12$, And $x+2y-2z=10$, And $x+y-z=4$ — are given to find x, y and z.

$$\left.\begin{array}{ll}\text{Suppose} & x+y+z=12\\ \text{And} & x+2y-2z=10\\ \text{And} & x+y-z=4\end{array}\right\}\text{ are given to find } x, y \text{ and } z.$$

From these three equations, two others may be derived which shall contain only *two* unknown quantities. One of the three unknown quantities in the original equations may be

QUEST.—How many independent equations does a problem of three or more unknown quantities furnish? What are independent equations? What are dependent ones? What identical ones?

exterminated, in the same manner as when there are at first only two, by the rules already given.

In the equations given above, if we transpose y and z, we shall have,

In the first, $x=12-y-z$.
In the second, $x=10-2y+2z$.
In the third, $x=4-y+z$.

From these we may deduce two new equations, from which x shall be excluded.

By making the 1st and 2d equal, $12-y-z=10-2y+2z$.
By making the 2d and 3d equal, $10-2y+2z=4-y+z$.
Reducing the first of these two, $y=3z-2$.
Reducing the second, $y=z+6$.

From these two equations one may be derived containing only *one* unknown quantity.

Making one equal to the other, $3z-2=z+6$
And, $z=4$. Hence,

285. To solve a problem containing *three* unknown quantities, and producing three independent equations.

First, from the three equations deduce two, containing only two unknown quantities.

Then, from these two deduce one, containing only one unknown quantity.

For making these reductions, the rules already given are sufficient. (Arts. 277, 279, 281.)

Prob. 38. Given $x+5y+6z=53$
2. And $x+3y+3z=30$
3. And $x+y+z=12$

$\left.\begin{array}{l} x+5y+6z=53 \\ x+3y+3z=30 \\ x+y+z=12 \end{array}\right\}$ To find x, y and z.

From these three equations to derive two, containing only two unknown quantities,

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QUEST.—Rule for solving problems with three unknown quantities?

4. Subtract the 2d from the 1st, $2y+3z=23$.

5. Subtract the 3d from the 2d, $2y+2z=18$.

From these two, to derive one,

6. Subtract the 5th from the 4th, $z=5$.

To find x and y, we have only to take their values from the third and fifth equations. (Art. 278.)

7. Reducing the fifth, $y=9-z=9-5=4$.

8. Transposing in the third, $x=12-z-y=12-5-4=3$.

Prob. 39. Given $\left.\begin{array}{ll} & x+y+z=12 \\ \text{And} & x+2y+3z=20 \\ \text{And} & \frac{1}{3}x+\frac{1}{2}y+z=6 \end{array}\right\}$ To find x, y and z.

286. In many of the examples in the preceding sections, the processes given might have been shortened. But the object has been to illustrate general principles, rather than to furnish specimens of expeditious solutions. The learner *will do well*, as he passes along, to exercise his skill in *abridging* the calculations which are here given, or *substituting others* in their stead.

Prob. 40. Given $\left.\begin{array}{ll} & x+y=a \\ \text{And} & x+z=b \\ \text{And} & y+z=c \end{array}\right\}$ To find x, y and z.

Prob. 41. Three persons, A, B and C, purchase a horse for 100 dollars, but neither is able to pay for the whole. The payment would require,

The whole of A's money, together with half of B's; or
The whole of B's, with one third of C's; or
The whole of C's, with one fourth of A's.
How much money had each?

287. The learner must exercise his own judgment, as to the choice of the quantity to be first exterminated. It will

QUEST.—How do you know which unknown quantity to exterminate first?

generally be best to begin with that which is most free from *co-efficients*, *fractions*, *radical signs*, &c.

Prob. 42. The sum of the distances which three persons, A, B and C, have travelled, is 62 miles; A's distance is equal to 4 times C's, added to twice B's; and twice A's added to 3 times B's, is equal to 17 times C's. What are the respective distances?

Prob. 43. $\left.\begin{array}{ll}\text{Given} & \frac{1}{2}x+\frac{1}{3}y+\frac{1}{4}z=62 \\ \text{And} & \frac{1}{3}x+\frac{1}{4}y+\frac{1}{5}z=47 \\ \text{And} & \frac{1}{4}x+\frac{1}{5}y+\frac{1}{6}z=38\end{array}\right\}$ To find x, y and z.

Prob. 44. $\left.\begin{array}{ll}\text{Given} & xy=600 \\ \text{And} & xz=300 \\ \text{And} & yz=200\end{array}\right\}$ To find x, y and z.

FOUR UNKNOWN QUANTITIES.

288. The same method which is employed for the reduction of three equations, may be extended to 4, 5, or any number of equations, containing as many unknown quantities.

The unknown quantities may be exterminated, one after another, and the number of equations may be reduced by successive steps from five to four, from four to three, from three to two, &c.

Prob. 45. To find w, x, y and z, from

$$\left.\begin{array}{lr}\text{1. The equation} & \frac{1}{2}y+z+\frac{1}{2}w=8 \\ \text{2. And} & x+y+w=9 \\ \text{3. And} & x+y+z=12 \\ \text{4. And} & x+w+z=10\end{array}\right\}\ \textit{Four}\text{ equations.}$$

$$\left.\begin{array}{lr}\text{5. Clear the 1st of frac.} & y+2z+w=16 \\ \text{6. Subtract 2d from 3d,} & z-w=3 \\ \text{7. Subtract 4th from 3d,} & y-w=2\end{array}\right\}\ \textit{Three}\text{ equations.}$$

QUEST.—How are problems solved containing four or five unknown quantities?

8. Adding 5th and 6th, $y+3z=19$ } *Two* equations.
9. Subtract 7th from 6th, $-y+z=1$ }
10. Adding 8th and 9th, $4z=20$. Or $z=5$ }
11. Transposing in the 8th, $y=19-3z=4$ } Quantities
12. Transposing in the 3d, $x=12-y-z=3$ } required.
13. Transposing in the 2d, $w=9-x-y=2$ }

Prob. 46. Given $w+50=x$
And $x+120=3y$
And $y+120=2z$
And $z+195=3w$
} To find w, x, y and z.

Answer. $w=100$ $y=90$
$=150$ $z=105$.

Prob. 47. There is a certain number consisting of two digits. The left hand digit is equal to three times the right hand digit; and if twelve be subtracted from the number itself, the remainder will be equal to the square of the left hand digit. What is the number? 93

Prob. 48. If a certain number be divided by the product of its two digits, the quotient will be 2; and if 27 be added to the number, the digits will be inverted. What is the number?

Prob. 49. There are two numbers, such, that if the less be taken from 3 times the greater, the remainder will be 35; and if 4 times the greater be divided by 3 times the less $+1$, the quotient will be equal to the less. What are the numbers?

Prob. 50. There is a certain fraction, such, that if 3 be added to the numerator, the value of the fraction will be $\frac{1}{3}$; but if 1 be subtracted from the denominator, the value will be $\frac{1}{5}$. What is the fraction?

Prob. 51. A gentleman has two horses, and a saddle which is worth ten guineas. If the saddle be put on the *first* horse, the value of both will be *double* that of the *second* horse; but if the saddle be put on the *second* horse, the value of both

will be less than that of the *first* horse by 13 guineas. What is the value of each horse?

Prob. 52. Divide the number 90 into 4 such parts, that the first *increased* by 2, the second *diminished* by 2, the third *multiplied* by 2, and the fourth *divided* by 2, shall all be equal.

If x, y and z, be three of the parts, the fourth will be $90-x-y-z$. And by the conditions, &c.

Prob. 53. Find three numbers, such that the *first* with $\frac{1}{2}$ the sum of the *second* and *third* shall be 120; the *second* with $\frac{1}{5}$ the difference of the *third* and *first* shall be 70; and $\frac{1}{2}$ the sum of the three numbers shall be 95.

Prob. 54. What two numbers are those, whose difference, sum and product, are as the numbers 2, 3 and 5?

Prob. 55. A vintner sold at one time, 20 dozen of port wine, and 30 dozen of sherry; and for the whole received 120 guineas. At another time, he sold 30 dozen of port and 25 dozen of sherry, at the same prices as before; and for the whole received 140 guineas. What was the price per dozen of each sort of wine?

Prob. 56. A merchant having mixed a certain number of gallons of brandy and water, found that, if he had mixed 18 gallons more of each, he would have put into the mixture 8 gallons of brandy for every 7 of water. But if he had mixed 18 less of each, he would have put in 5 gallons of brandy for every 4 of water. How many gallons of each did he mix?

Prob. 57. What fraction is that, whose numerator being doubled, and the denominator increased by 7, the value becomes $\frac{2}{3}$; but the denominator being doubled, and the numerator increased by 2, the value becomes $\frac{3}{5}$?

Prob. 58. A lad expends 30 cents in apples and pears, giving a cent for 4 apples and a cent for 5 pears. He after-

wards parts with half his apples and one third of his pears, the cost of which was 13 cents. How many did he buy of each? 72 & 66.

289. If in the algebraic statement of the conditions of a problem, the original equations are more numerous than the unknown quantities; these equations will either be *contradictory*, or one or more of them will be *superfluous*.

Thus the equations $3x=60$
And $\frac{1}{2}x=20$ $\Big\}$ are contradictory.

For by the first $x=20$, while by the second, $x=40$

But if the latter equation be altered, so as to give to x the same value as in the former, it will be useless, in the statement of a problem. For nothing can be determined from the one, which cannot be from the other.

Thus of the equations $3x=60$
And $\frac{1}{2}x=10$ $\Big\}$ one is superfluous.

290. But if the number of independent equations produced from the conditions of a problem, is *less* than the number of unknown quantities, the subject is not sufficiently limited to admit of a definite answer. If for instance, in the equation $x+y=100$, x and y are required, there may be fifty different answers. The values of x and y may be either 99 and 1, or 98 and 2, or 97 and 3, &c. For the sum of each pair of these numbers is equal to 100. But if there is a second equation which determines *one* of these quantities, the other may then be found from the equation already given. As $x+y=100$, if $x=46$, y must be such a number as added to 46 will make 100, that is, it must be 54. No other number will answer this condition.

QUEST.—When the equations are more numerous than the unknown quantities, what is said of them?

291. For the sake of abridging the solution of a problem, however, the number of independent equations actually put upon paper is frequently less than the number of unknown quantities.

Prob. 59. To find two numbers whose sum is 30, and the difference of their squares 120.

292. In most cases also, the solution of a problem which contains many unknown quantities, may be abridged by particular artifices in *substituting* a single letter for several.

Prob. 60. Suppose four numbers, u, x, y and z, are required, of which the sum of the first three is 13, the sum of the first two and last 17, the sum of the first and last two 18, the sum of the last three 21.

Then 1. $u+x+y=13$
2 $u+x+z=17$
3 $u+y+z=18$
4 $x+y+z=21.$

Let S be substituted for the *sum* of the four numbers, that is, for $u+x+y+z$. (Art. 159.) It will be seen that of these four equations,

The first contains all the letters except z, that is $S-z=13$
The second contains all except y, that is, $S-y=17$
The third contains all except x, that is, $S-x=18$
The fourth contains all except u, that is, $S-u=21$

Adding all these equations together, we have

$$4S-z-y-x-u=69$$

Or $4S-(z+y+x+u)=69$. (Art. 67.)
But $S=(z+y+x+u)$ by substitution.

Therefore, $4S-S=69$, that is, $3S=69$, and $S=23$.

Then putting 23 for S, in the four equations in which it is first introduced, we have

$$\left.\begin{array}{r} 23-z=13 \\ 23-y=17 \\ 23-x=18 \\ 23-u=21 \end{array}\right\} \text{Therefore} \left\{\begin{array}{l} z=23-13=10 \\ y=23-17=6 \\ x=23-18=5 \\ u=23-21=2 \end{array}\right.$$

N. B. Contrivances of this sort for facilitating the solution of particular problems, must be left to be furnished for the occasion by the teacher and the ingenuity of the learner. They are of a nature not to be taught by a system of rules.

SECTION XII

RATIO AND PROPORTION.

ART. 293. The design of mathematical investigations, is to arrive at the knowledge of particular quantities, by comparing them with other quantities, either *equal* to, or *greater*, or *less* than those which are the objects of inquiry. This end is most commonly attained by means of a series of *equations* and *proportions*. When we make use of equations, we determine the quantity sought, by discovering its *equality* with some other quantity or quantities already known.

We have frequent occasion, however, to compare the unknown quantity with others which are *not equal* to it, but either greater or less.

294. *Unequal quantities* may be compared with each other in two ways.

QUEST.—What is the design of mathematical investigations? How is this end commonly attained? In equations how is the value of the unknown quantity determined? In how many ways are unequal quantities compared? What are they?

First, We may inquire *how much* one of the quantities is greater than the other: or,

Second, We may inquire *how many times* one quantity contains the other.

295. The *relation* which is found to exist between the two quantities compared, is called the *ratio* of the two quantities.

RATIO *is of two kinds, arithmetical and geometrical.* It is also sometimes called, *ratio by subtraction*, and *ratio by division*.

296. ARITHMETICAL RATIO *is the* DIFFERENCE *between two quantities or sets of quantities.* The quantities themselves are called the *terms* of the ratio, that is, the terms between which the ratio exists. Thus 2 is the arithmetical ratio of 5 to 3. This is sometimes expressed, by placing two points between the quantities thus, 5 ·· 3, which is the same as 5—3. Indeed the term arithmetical ratio, and its notation by points, are almost needless, and are seldom used. For the one is only a substitute for the word *difference* and the other for the sign —.

297. If both the terms of an arithmetical ratio be *multiplied* or *divided* by the same quantity, the *ratio* will, in effect, be multiplied or divided by that quantity.

Thus if $a-b=r$

Then multiply both sides by h, (Ax. 3,) $ha-hb=hr$

And dividing by h, (Ax. 4,) $\frac{a}{h}-\frac{b}{h}=\frac{r}{h}$.

298. If the terms of one arithmetical ratio be added to, or subtracted from, the corresponding terms of another, the ratio

QUEST.—What is ratio? Of how many kinds is it? What are they called? What is arithmetical ratio? What are the quantities themselves called? If both the terms are *multiplied*, or *divided*, by the same quantity, how is the ratio affected? If the terms of one ratio are added to the corresponding terms of another, how is the ratio affected?

of their sum or difference will be equal to the sum or difference of the two ratios.

If $\left.\begin{array}{l} a-b \\ \text{And } d-h \end{array}\right\}$ are the two ratios,

Then $(a+d)-(b+h)=(a-b)+(d-h)$. For each $=a+d-b-h$.
And $(a-d)-(b-h)=(a-b)-(d-h)$. For each $=a-d-b+h$.

Thus the arithmetical ratio of $11 .. 4$ is 7,
And the arithmetical ratio of $5 .. 2$ is 3.

The ratio of the *sum* of the terms $16 .. 6$ is 10, which is also the sum of the ratios 7 and 3.

The ratio of the *difference* of the terms $6 .. 2$ is 4, which is also the difference of the ratios 7 and 3.

299. GEOMETRICAL RATIO *is that relation between quantities which is expressed by the* QUOTIENT *of the one divided by the other.*

Thus the ratio of 8 to 4, is $\frac{8}{4}$ or 2. For this is the quotient of 8 divided by 4. In other words, it shows how often 4 is contained in 8. So $a:b$ expresses the ratio of a to b.

300. The two quantities compared, are called a *couplet.* The *first* term is the *antecedent*, and the *last*, the *consequent.*

301. GEOMETRICAL RATIO is expressed in *two ways.*

First, In the form of a *fraction*, making the *antecedent* the *numerator*, and the *consequent* the *denominator*; thus the ratio of a to b is $\frac{a}{b}$. And

Second, By placing a *colon* between the quantities compared; thus, $a:b$ expresses the ratio of a to b.

Obser. The French mathematicians put the *antecedent* for the *denominator*; and the *consequent* for the *numerator*. Some American

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QUEST.—What is geometrical ratio? What is a couplet? The antecedent? The consequent? In how many ways is geometrical ratio expressed? The first? Second? What is the French mode? What are the comparative advantages of the English and French methods?

authors have followed their example. It is believed however that the English method, which is adopted in the larger work, is most in accordance with reason; while the French mode may perhaps have some advantage in practice.

302. Of these three, the antecedent, the consequent, and the ratio, any *two* being given, the other may be found.

Let $a=$ the antecedent, $c=$ the consequent, $r=$ the ratio.

By definition $r=\frac{a}{c}$; that is, the ratio is equal to the antecedent divided by the consequent.

Multiplying by c, $a=cr$, that is, the antecedent is equal to the consequent multiplied into the ratio.

Dividing by r, $c=\frac{a}{r}$, that is, the consequent is equal to the antecedent divided by the ratio.

Cor. 1. If two couplets have their antecedents equal, and their consequents equal, their ratios must be equal. (Euc. 7. 5.)

Cor. 2. If in two couplets, the ratios are equal, and the antecedents equal, the consequents are equal; and if the ratios are equal and the consequents equal, the antecedents are equal. (Euclid 9. 5.)

303. If the two quantities compared are *equal*, the ratio is a unit, *or a ratio of equality.* The ratio of $3\times6 : 18$ is a unit, for the quotient of any quantity divided by itself is 1.

If the antecedent of a couplet is *greater* than the consequent, the ratio is greater than a unit. For if a dividend is greater than its divisor, the quotient is greater than a unit.

Quest.—When the antecedent and consequent are given, how is the ratio found? When the consequent and ratio are given, how find the antecedent? When the antecedent and ratio are given, how find the consequent? What is the first corollary? The second? If the two quantities compared are equal, what is the ratio? If the antecedent is the largest, what is the ratio? What called?

Thus the ratio of 18 : 6 is 3. (Art. 104, Cor.) This is called a ratio of *greater inequality.*

On the other hand, if the antecedent is *less* than the consequent, the ratio is less than a unit, and is called a ratio of *less inequality.* Thus the ratio of 2 : 3, is less than a unit, because the dividend is less than the divisor.

305. INVERSE *or* RECIPROCAL *ratio is the ratio of the reciprocals of two quantities.* (Art. 92.)

Thus the reciprocal ratio of 6 to 3, is $\frac{1}{6}$ to $\frac{1}{3}$, that is $\frac{1}{6} \div \frac{1}{3}$.

The direct ratio of a to b is $\frac{a}{b}$, that is, the antecedent divided by the consequent.

The reciprocal ratio is $\frac{1}{a} : \frac{1}{b}$, or, $\frac{1}{a} \div \frac{1}{b} = \frac{1}{a} \times \frac{b}{1} = \frac{b}{a}$;

that is, the consequent b divided by the antecedent a.

Hence a reciprocal ratio is expressed by *inverting the fraction* which expresses the direct ratio; or when the notation is by points, by *inverting the order of the terms.*

Thus a is to b inversely, as b to a.

306. COMPOUND RATIO *is the ratio of the* PRODUCTS *of the corresponding terms of two or more simple ratios.*

Thus the ratio of	6 : 3, is 2
And the ratio of	12 : 4, is 3
The ratio compounded of these is	72 : 12=6

Here the compound ratio is obtained by multiplying together the two antecedents, and also the two consequents, of the simple ratios. Hence it is equal to the product of the simple ratios.

QUEST.—If the consequent is the largest, what is the ratio? What called? What is inverse ratio? How expressed? What is compound ratio? Does it differ from other ratio in its nature?

Compound ratio is not different in its *nature* from any other ratio. The term is used to denote the origin of the ratio in particular cases.

307. If in a series of ratios the consequent of each preceding couplet, is the antecedent of the following one, *the ratio of the first antecedent to the last consequent, is equal to that which is compounded of all the intervening ratios.* (Euclid 5th B.)

Thus, in the series of ratios $a:b$
$b:c$
$c:d$
$d:h$

the ratio of $a:h$, is equal to that which is compounded of the ratios of $a:b$, of $b:c$, of $c:d$, of $d:h$. For the compound ratio by the last article is $\frac{abcd}{bcdh}=\frac{a}{h}$ or $a:h$. (Art. 117.)

308. A particular class of compound ratios is produced, by multiplying a simple ratio into *itself*, or into another *equal* ratio. These are termed *duplicate, triplicate, quadruplicate*, &c., according to the number of multiplications.

A ratio compounded of *two* equal ratios, that is, the *square* of the simple ratio, is called a *duplicate* ratio.

One compound of *three*, that is, the *cube* of the simple ratio, is called *triplicate*, &c.

In a similar manner, the ratio of the *square roots* of two quantities, is called a *subduplicate* ratio; that of the *cube roots* a *subtriplicate* ratio, &c.

Thus the simple ratio of a to b, is $a:b$

The duplicate ratio of a to b, is $a^2:b^2$

QUEST.—What is it equal to? When the consequent of each preceding couplet is the antecedent of the next, what is the ratio of the first antecedent to the last consequent equal to? What is a duplicate ratio? Triplicate? Subduplicate? Subtriplicate?

The triplicate ratio of a to b, is $a^3 : b^3$

The subduplicate ratio of a to b, is $\sqrt{a} : \sqrt{b}$

The subtriplicate ratio of a to b, is $\sqrt[3]{a} : \sqrt[3]{b}$, &c.

N. B. The terms *duplicate*, *triplicate*, &c., must not be confounded with *double*, *triple*, &c.

The ratio of 6 to 2 is	$6:2=3$
Double this ratio, that is, *twice* the ratio, is	$12:2=6$
Triple the ratio, i. e. *three times* the ratio, is	$18:2=9$
The *duplicate* ratio, i. e. the *square* of the ratio, is	$6^2:2^2=9$
The *triplicate* ratio, i. e. the *cube* of the ratio, is	$6^3:2^3=27$

309. That quantities may have a ratio to each other, it is necessary that they should be so far of the same nature, that one can properly be said to be either equal to, or greater, or less than the other. Thus a foot has a ratio to an inch, for one is twelve times as great as the other.

310. From the mode of expressing *geometrical ratios* in the *form* of a *fraction*, (Art. 301,) it is obvious that the *ratio* of two quantities is the same as the *value* of a fraction whose numerator and denominator are equal to the antecedent and consequent of the given ratio. Hence,

311. *To multiply, or divide both the antecedent and consequent by the same quantity, does not alter the ratio.* (Art. 112.) *To multiply, or divide the antecedent alone by any quantity, multiplies or divides the ratio; to multiply the consequent alone, divides the ratio; and to divide the consequent, multiplies the ratio.* (Arts. 132, 135.) That is, *multiplying* and *dividing* the antecedent or consequent, has the *same* effect on the *ratio*, as a similar operation, performed on the numerator or denominator, has upon the *value* of a fraction.

QUEST.—What effect does it have on the ratio to multiply or divide both the antecedent and the consequent by the same quantity? To multiply or divide the antecedent only? The consequent only?

312. *If to or from the terms of any couplet, two other quantities having the same ratio be added or subtracted, the sums or remainders will also have the same ratio.* (Euclid 5 and 6. 5.) Thus the ratio of 12:3 is the same as that of 20:5. And the *ratio* of the *sum* of the *antecedents* 12+20 to the *sum* of the *consequents* 3+5, is the same as the ratio of either couplet. That is,

$$12+20:3+5::12:3=20:4, \text{ or } \frac{12+20}{3+5}=\frac{12}{3}=\frac{20}{5}=4.$$

So also the *ratio* of the *difference* of the *antecedents*, to the *difference* of the *consequents*, is the same. That is,

$$20-12:5-3::12:3=20:5, \text{ or } \frac{20-12}{5-3}=\frac{12}{3}=\frac{20}{5}=4.$$

313. If in several couplets the ratios are equal, *the sum of all the antecedents has the same ratio to the sum of all the consequents, which any one of the antecedents has to its consequent.* (Euclid 1 and 2. 5.)

$$\text{Thus the ratio} \begin{cases} 12:6=2 \\ 10:5=2 \\ 8:4=2 \\ 6:3=2 \end{cases}$$

Therefore the ratio of $(12+10+8+6):(6+5+4+3)=2$.

EXAMPLES FOR PRACTICE.

1. Which is the greater, the ratio of 11:9, or that of 44:35?

2. Which is the greater, the ratio of $a+3:\frac{1}{6}a$, or that of $2a+7:\frac{1}{3}a$?

QUEST.—When you add or subtract the terms of two couplets having the same ratio, what is the ratio of their sum or difference? In several couplets of equal ratios, what ratio has the sum of all the antecedents to the sum of all the consequents?

3. If the antecedent of a couplet be 65, and the ratio 13, what is the consequent?

4. If the consequent of a couplet be 7, and the ratio 18, what is the antecedent?

5. What is the ratio compounded of the ratios of $3:7$, and $2a:5b$, and $7x+1:3y-2$?

6. What is the ratio compounded of $x+y:b$, and $x-y:a+b$, and $a+b:h$?

7. If the ratios of $5x+7:2x-3$, and $x+2:\frac{1}{2}x+3$ be compounded, will they produce a ratio of greater inequality, or of less inequality?

8. What is the ratio compounded of $x+y:a$, and $x-y:b$, and $b:\dfrac{x^2-y^2}{a}$?

9. What is the ratio compounded of $7:5$, and the duplicate ratio of $4:9$, and the triplicate ratio of $3:2$?

10. What is the ratio compounded of $3:7$, and the triplicate ratio of $x:y$, and the subduplicate ratio of $49:9$?

PROPORTION.

315. PROPORTION *is an equality of ratios. It is divided into two kinds: Arithmetical and Geometrical.*

Arithmetical proportion is an equality of arithmetical ratios, and *geometrical* proportion is an equality of geometrical ratios. Thus the numbers 6, 4, 10, 8, are in *arithmetical* proportion, because the *difference* between 6 and 4 is the same as the difference between 10 and 8. And the numbers 6, 2, 12, 4, are in *geometrical* proportion, because the *quotient* of 6 divided by 2, is the same as the quotient of 12 divided by 4.

QUEST.—What is proportion? Of how many kinds is it? What is arithmetical proportion? Geometrical proportion?

316. Care must be taken not to confound *proportion* with *ratio*. This caution is the more necessary, as in common discourse, the two terms are used indiscriminately, or rather, proportion is used for both. The expenses of one man are said to bear a greater proportion to his income, than those of another. But according to the definition which has just been given, one *proportion* is neither greater nor less than another. For *equality* does not admit of degrees. One *ratio* may be greater or less than another. The ratio of 12 : 2 is greater than that of 6 : 2, and less than that of 20 : 2. But these differences are not applicable to *proportion*, when the term is used in its technical sense. The loose signification which is so frequently attached to this word, may be proper enough in *familiar language*; for it is sanctioned by general usage. But for scientific purposes, the *distinction* between *proportion* and *ratio* should be clearly drawn, and cautiously observed.

317. Proportion may be expressed, either by the common sign of equality, or by four points between the two couplets.

Thus $\left\{\begin{array}{l} 8 \cdot\cdot 6 = 4 \cdot\cdot 2, \text{ or } 8 \cdot\cdot 6 :: 4 \cdot\cdot 2 \\ a \cdot\cdot b = c \cdot\cdot d, \text{ or } a \cdot\cdot b :: c \cdot\cdot d \end{array}\right\}$ are arithmetical proportions.

And $\left\{\begin{array}{l} 12 : 6 = 8 : 4, \text{ or } 12 : 6 :: 8 : 4 \\ a : b = d : h, \text{ or } a : b :: d : h \end{array}\right\}$ are geometrical proportions.

The latter is read, 'the ratio of a to b equals the ratio of d to h;' or more concisely, 'a is to b as d to h.'

318. The first and last terms are called the *extremes*, and the other two the *means*. *Homologous* terms are either the two antecedents or the two consequents. *Analogous* terms are the antecedent and consequent of the same couplet.

Quest.—What is the difference between *ratio* and *proportion*? In how many ways is proportion expressed? How is the latter read? Which are the extremes? Which the means? What are homologous terms? What analogous terms?

319. As the ratios are equal, it is manifestly immaterial which of the two couplets is placed first.

If $a:b::c:d$, then $c:d::a:b$. For if $\frac{a}{b}=\frac{c}{d}$ then $\frac{c}{d}=\frac{a}{b}$.

320. The ***number* of *terms*** in a proportion must be at least *four*. For the equality is between the ratios of *two couplets*; and each couplet must have an antecedent and a consequent. There may be a proportion, however, among *three quantities*. For one of the quantities may be *repeated*, so as to form two terms. In this case the quantity repeated is called the *middle term*, or a ***mean proportional*** between the two other quantities, especially if the proportion is geometrical.

Thus the numbers 8, 4, 2, are proportional. That is, $8:4::4:2$. Here 4 is both the consequent in the first couplet, and the antecedent in the last. It is therefore a mean proportional between 8 and 2.

The *last* term is called a *third proportional* to the two other quantities. Thus 2 is a third proportional to 8 and 4.

321. *Inverse* or *reciprocal* proportion is an equality between a *direct* ratio and a *reciprocal* ratio.

Thus $4:2::\frac{1}{3}:\frac{1}{6}$; that is, 4 is to 2 *reciprocally*, as 3 to 6. Sometimes also, the order of the terms in one of the couplets is inverted, without writing them in the form of a fraction. (Art. 305.)

Thus $4:2::3:6$ inversely. In this case, the *first* term is to the *second*, as the *fourth* to the *third*; that is, the first divided by the second, is equal to the fourth divided by the third.

322. When there is a series of quantities, such that the ratios of the first to the second, of the second to the third, of

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QUEST.—Which couplet must be placed first? How many terms must there be? Can there be a proportion with three quantities? What is the middle term called? The last term? What is inverse proportion?

the third to the fourth, &c., are *all equal*; the quantities are said to be in *continued proportion*. The consequent of each preceding ratio is then the antecedent of the following one.

N. B. Continued proportion is also called *progression*.

323. In the preceding articles of this section, the general properties of ratio and proportion have been defined and illustrated. It now remains to consider the principles which are peculiar to each kind of proportion, and attend to their practical application in the solution of problems.

SECTION XIII.

ARITHMETICAL PROPORTION AND PROGRESSION.

Art. 324. If *four* quantities are in arithmetical proportion, *the sum of the extremes is equal to the sum of the means.*

Thus if $a \cdot\cdot b :: h \cdot\cdot m$, then $a+m=b+h$

For by supposition, $a-b=h-m$

And transposing $-b$ and $-m$, $a+m=b+h$.

So in the proportion, $12 \cdot\cdot 10 :: 11 \cdot\cdot 9$, we have $12+9=10+11$.

325. Again if *three* quantities are in arithmetical proportion, *the sum of the extremes is equal to double the mean.*

If $a \cdot\cdot b :: b \cdot\cdot c$, then, $a-b=b-c$

And transposing $-b$ and $-c$, $a+c=2b$.

326. Quantities, which *increase* by a common difference, as 2, 4, 6, 8, 10, &c.; or *decrease* by a common difference, as

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Quest.—When four quantities are in arithmetical proportion, what is the sum of the extremes equal to? When there are but three terms in the proportion, what is the sum of the extremes equal to? What is continued arithmetical proportion?

15, 12, 9, 6, 3, &c., are in *continued arithmetical proportion.* (Art. 322.)

Such a series is also called an *arithmetical progression;* and sometimes *progression by difference,* or *equidifferent series.*

327. When the quantities *increase,* they form what is called an *ascending* series, as 3, 5, 7, 9, 11, &c.

When they *decrease,* they form a *descending* series, as 11, 9, 7, 5, 3, &c.

The natural numbers, 1, 2, 3, 4, 5, 6, &c., are in arithmetical progression ascending.

328. From the definition it is evident that in an *ascending* series, each succeeding term is found, by *adding the common difference* to the preceding term.

If the first term is 3, and the common difference 2;

The series is 3, 5, 7, 9, 11, 13, &c.

If the first term is a, and the common difference d;

Then $a+d$ is the second term, $a+d+d=a+2d$ the third, $a+2d+d=a+3d$ the 4th, $a+3d+d=a+4d$ the 5th, &c.

And the series is $\overset{1st}{a}$, $\overset{2d}{a+d}$, $\overset{3d}{a+2d}$, $\overset{4th}{a+3d}$, $\overset{5th}{a+4d}$, &c.

If the first term and the common difference are the *same,* the series becomes more simple. Thus if a is the first term, and also the common difference, and n the number of terms,

Then $a+a=2a$, is the second term,

$2a+a=3a$, the third, &c.

And the series is a, $2a$, $3a$, $4a$, ... na.

329. In a *descending* series, each succeeding term is found by *subtracting the common difference* from the preceding term.

QUEST.—What else is this series called? When the series increases, what is it called? When it decreases, what? How is each successive term found in an ascending series? How in a descending series?

If a is the first term, and d the common difference, the series is $\overset{1st}{a}$, $\overset{2d}{a-d}$, $\overset{3d}{a-2d}$, $\overset{4th}{a-3d}$, $\overset{5th}{a-4d}$, &c.

In this manner we may obtain any term by continued addition or subtraction. But in a long series, this process would become tedious. There is a method much more expeditious. By attending to the series,

$$\overset{1st}{a},\ \overset{2d}{a+d},\ \overset{3d}{a+2d},\ \overset{4th}{a+3d},\ \overset{5th}{a+4d},\ \&c.,$$

it will be seen that the number of times d is added to a, is *one less* than the number of the term. Thus,

The *second* term is $a+d$, i. e. a added to *once* d;
The *third* is $a+2d$, a added to *twice* d;
The *fourth* is $a+3d$, a added to *thrice* d; &c.
So if the series be continued,
The 50th term will be $a+49d$,
The 100th term $a+99d$

If the series be *descending*, the 100th term will be $a-99d$.
In the *last* term, the number of times d is added to a, is *one less* than the number of all the terms. If then
d=the common difference, a=the first term, z=the last, n=the number of terms, we shall have in all cases,

$$z=a\pm(n-1)\times d;\ \text{that is,}$$

330. (1.) To find the last term of an ascending series,
Add to the first term the product of the common difference into the number of terms less one, and the sum will be the last term.

(2.) To find the last term of a descending series.
From the first term subtract the product of the common difference into the number of terms less one, and the remainder will be the last term.

QUEST.—How is the last term of an ascending series found? How the last of a descending series?

N. B. Any other term may be found in the same way. For the series may be made to stop at any term, and that may be considered, for the time, as the last.

Thus the mth term$=a\pm(m-1)\times d$.

Prob. 1. If the first term of an ascending series is 7, the common difference 3, and the number of terms 9, what is the last term? Ans. $z=a+(n-1)d=7+(9-1)\times3=31$.

Prob. 2. If the first term of a descending series is 60, the common difference 5, and the number of terms 12, what is the last term? Ans. $z=a-(n-1)d=60-(12-1)\times5=5$.

Prob. 3. If the first term of an ascending series be 9, and the common difference 4, what will the 5th term be?

Ans. $z=a+(m-1)\times d=9+(5-1)\times4=25$.

331. There is one other inquiry to be made concerning a series in arithmetical progression. It is often necessary to find the *sum of all the terms*. This is called the *summation* of the series. The most obvious mode of obtaining the amount of the terms, is to add them together. But the nature of progression will furnish us with a more expeditious method.

Let us take, for instance, the series	3,	5,	7,	9,	11,
And also the same inverted,	11,	9,	7,	5,	3,
The sums of the terms will be,	14,	14,	14,	14,	14.
Take also the series,	a	$a+d$,	$a+2d$,	$a+3d$,	$a+4d$
And the same inverted,	$a+4d$,	$a+3d$,	$a+2d$,	$a+d$,	a
The sums will be,	$2a+4d$,	$2a+4d$,	$2a+4d$,	$2a+4d$,	$2a+4d$

Hence, it will be perceived that the sum of *all* the terms in the double series, is equal to the sum of the extremes repeated as many times as there are terms. Thus,

The sum of 14, 14, 14, 14 and $14=14\times5$.

And the sum of the terms in the other double series is $(2a+4d)\times 5$.

But this is *twice* the sum of the terms in the *single* series. If then we put

a=the first term, n=the number of terms,

z=the last, s=the sum of the terms,

we shall have this equation, $s=\frac{a+z}{2}\times n$. Hence,

332. To find the sum of all the terms in an arithmetical progression.

Multiply half the sum of the extremes into the number of terms, and the product will be the sum of the given series.

Prob. 4. What is the sum of the natural series of numbers 1, 2, 3, 4, 5, &c., up to 1000?

$$\text{Ans. } s=\frac{a+z}{2}\times n=\frac{1+1000}{2}\times 1000=500500.$$

333. The two formulas, $z=a\pm(n-1)d$, (Art. 329,) and $s=\frac{a+z}{2}\times n$, (Art. 331,) contain five different quantities; viz. *a, the first term; d, the common difference; n, the number of terms; z, the last term; and s, the sum of all the terms.*

From these two formulas others may be deduced by which, if any *three* of the *five* quantities are given, the remaining two may easily be found. The most useful of these formulas are the following.

By the first formula,

1. *The last term*, $z=a\pm(n-1)d$, in which a, n and d are given.

QUEST.—How is the sum of all the terms found? When the first term, the common difference, and the number of terms are given, how is the last term found?

Transposing $(n-1)d$,

2. *The first term*, $a=z\pm(n-1)d$, z, n and d being given.

Transposing a in the 1st, and dividing by $n-1$,

3. *The common difference*, $d=\dfrac{z-a}{n-1}$, a, z and n being given.

Transposing and dividing,

4. *The number of terms*, $n=\dfrac{z-a}{d}+1$, a, z and d being given.

By the second formula,

5. *The sum of the terms*, $s=\dfrac{a+z}{2}\times n$, a, z and n being given.

Or by substituting for z its value,

$s=\dfrac{2a+(n-1)d}{2}\times n$, in which a, n and d are given.

Reducing the preceding equation,

6. *The first term*, $a=\dfrac{2s-dn^2+dn}{2n}$, s, d and n being given.

7. *The common difference*, $d=\dfrac{2s-2an}{n^2-n}$, s, a and n being given.

8. *The number of terms* $n=\dfrac{\sqrt{(2a-d)^2+8ds}-2a+d}{2d}$, a, d and s being given.

QUEST.—When the last term, the common difference, and the number of terms are given, how find the first term? When the first, and last, and the number of terms are given, how find the common difference? When the first and last terms, and the common difference are given, how find the number of terms? When the first, and last, and the number of terms are given, how find the sum of all the terms? When the sum, difference, and number of terms are given, how find the first term? When the first term, the sum, and the number of terms are given, how find the common difference? When the first term, the common difference, and the sum of the terms are given, how find the number of terms?

Obser. A variety of other formulas may be deduced from the preceding equations, the investigation of which will afford the student a pleasing and profitable exercise.

334. By the third formula, e. g. may be found any number of arithmetical means, between two given numbers. For the whole number of terms consists of the *two extremes* and all the *intermediate* terms. If then $m=$ number of means, $m+2=n$, the whole number of terms. Substituting $m+2$ for n in the third equation, we have,

The common difference, $d=\frac{z-a}{m+1}$, in which a, z and m are given.

Prob. 5. Find 6 arithmetical means, between 1 and 43.

Ans. { The common difference is 6,
The series, 1, 7, 13, 19, 25, 31, 37, 43.

335. It is obvious from the illustration in Art. 331, that *the sum of the extremes* in an arithmetical progression, *is equal to the sum of any other two terms equally distant from the extremes.* Thus, in the series 3, 5, 7, 9, 11, the sum of the first and last terms, of the first but one and last but one, &c., is the same in each case, viz. 14. The same is true of every series.

Prob. 6. If the first term of an increasing arithmetical series is 3, the common difference 2, and the number of terms 20; what is the sum of the series?

Prob. 7. If 100 stones are placed in a straight line, at the distance of a yard from each other; how far must a person travel, to bring them one by one to a box placed at the distance of a yard from the first stone?

QUEST.—How find any number of arithmetical means between two given numbers? In a series of arithmetical progression, what is the sum of the extremes equal to?

Prob. 8. What is the sum of 150 terms of the series

$$\frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, 2, \frac{7}{3}, \&c.?$$

Prob. 9. If the sum of an arithmetical series is 1455, the least term 5, and the number of terms 30; what is the common difference?

Prob. 10. If the sum of an arithmetical series is 567, the first term 7, and the common difference 2; what is the number of terms?

Prob. 11. What is the sum of 32 terms of the series

$1, 1\frac{1}{2}, 2, 2\frac{1}{2}, 3, \&c.?$

Prob. 12. A gentleman bought 47 books, and gave 10 cents for the first, 30 cents for the second, 50 cents for the third, &c. What did he give for the whole?

Prob. 13. A person put into a charity box, a cent the first day of the year, two cents the second day, three cents the third day, &c., to the end of the year. What was the whole sum for 365 days?

Prob. 14. How many strokes does a common clock strike in 24 hours?

Prob. 15. The clocks of Venice go on to 24 o'clock; how many strokes do they strike in a day?

Prob. 16. Required the sum of the odd numbers 1, 3, 5, 7, 9, &c. continued to 101 terms.

Prob. 17. Required the 365th term of the series of even numbers 2, 4, 6, 8, 10, 12, &c.

Prob. 18. The first term of a series is 4, the common difference 3, and the number of terms 100; what is the last term?

Prob. 19. A man puts $1 at interest at 6 per cent.; what will be the amount in 40 years?

Prob. 20. The extremes of a series are 2 and 29; and the number of terms is ten. What is the common difference? 3

Prob. 21. The extremes of a series are 3 and 39, and the common difference 2. What is the number of terms?

Prob. 22. Find 5 means between 6 and 48.

Prob. 23. Find 6 means between 8 and 36.

336. Problems of various kinds, in arithmetical progression, may be solved by stating the conditions algebraically, and then reducing the equations. Thus,

Prob. 24. Find four numbers in arithmetical progression, whose sum shall be 56, and the sum of their squares 864.

If $x=$ the second of the four numbers,
And $y=$ their common difference:

The series will be $x-y$, x, $x+y$ and $x+2y$.

By the conditions, $(x-y)+x+(x+y)+(x+2y)=56$

And $(x-y)^2+x^2+(x+y)^2+(x+2y)^2=864$

That is, $4x+2y=56$

And $4x^2+4xy+6y^2=864$

Reducing these equations, we have $x=12$, and $y=4$.

The numbers required, therefore, are 8, 12, 16 and 20.

Prob. 25. The sum of three numbers in arithmetical progression is 9, and the sum of their cubes is 153. What are the numbers? 1, 3, 5,

Prob. 26. The sum of three numbers in arithmetical progression is 15, and the sum of the squares of the two extremes is 58. What are the numbers? 3, 5, 7,

Prob. 27. There are four numbers in arithmetical progression: the sum of the squares of the first two is 34; and the sum of the squares of the last two is 130. What are the numbers?

$y = 5$

Prob. 28. A certain number consists of three digits, which are in arithmetical progression, and the number divided by the sum of its digits is equal to 26; but if 198 be added to it, the digits will be inverted. What is the number?

Let the digits be equal to $x-y$, x, and $x+y$, respectively. Then the number$=100(x-y)+10x+(x+y)=111x-99y$, &c.

Prob. 29. The sum of the squares of the extremes of four numbers in arithmetical progression is 200; and the sum of the squares of the means is 136. What are the numbers?

Prob. 30. There are four numbers in arithmetical progression whose sum is 28, and their continued product is 585. What are the numbers?

SECTION XIV.

GEOMETRICAL PROPORTION AND PROGRESSION.

Art. 337. If four quantities are in *geometrical proportion, the product of the extremes is equal to the product of the means.* Thus,

$12:8::15:10$; therefore $12\times10=8\times15$. Hence,

338. Any *factor* may be transferred from one of the means to the other, or from one extreme to the other, without affecting the proportion.

Thus if $a:mb::x:y$, then $a:b::mx:y$; for the product of the means in both cases is the same.

So if $na:b::x:y$, then $a:b::x:ny$.

Quest.—If four quantities are in geometrical proportion, what is the product of the extremes equal to?

339. On the other hand, if the product of two quantities is equal to the product of two others, the four quantities will form a proportion if they are so arranged, that those on one side of the equation shall constitute the means, and those on the other side the extremes. Thus since $6\times12=8\times9$, then $6:8::9:12$. (Art. 158.)

Cor. The same must be true of *any factors* which form the two sides of an equation. Thus if

$$(a+b)\times c=(d-m)\times y,\text{ then } a+b:d-m::y:c.$$

340. If *three* quantities are proportional, the product of the extremes is equal to the *square* of the mean. For this mean proportional is, at the same time, the consequent of the first couplet, and the antecedent of the last. (Art. 320.) It is therefore to be multiplied *into itself*, that is, it is to be *squared*.

Thus, $4:6::6:9$; therefore $4\times9=6\times6$.

If $a:b::b:c$, then mult. extremes and means, $ac=b^2$.

Hence, a *mean proportional* between two quantities may be found by *extracting the square root of their product.*

If $a:x::x:c$, then $x^2=ac$, and $x=\sqrt{ac}$. (Art. 249.)

341. It follows, from Art. 338, that in proportion, either extreme is equal to the product of the means, divided by the other extreme; and either of the means is equal to the product of the extremes, divided by the other mean.

1. If $a:b::c:d$, then	$ad=bc$.
2. Dividing by d,	$a=bc\div d$.
3. Dividing the first by c,	$b=ad\div c$.
4. Dividing it by b,	$c=ad\div b$.
5. Dividing it by a,	$d=bc\div a$.

QUEST.—How is an equation put into a proportion? If three quantities are in proportion, what is the product of extremes equal to? How is a mean proportional between two quantities found? When the means and one extreme are given, how find the other extreme? When the extremes and one of the means are given, how find the other?

That is, the *fourth* term is equal to the *product of the second and third divided by the first.*

N. B. On this principle is founded the rule of simple proportion in arithmetic, commonly called the "*Rule of Three.*" Three numbers are given to find a fourth, which is obtained by multiplying together the second and third, and dividing by the first.

342. The propositions respecting the products of the means and of the extremes, furnish a very simple and convenient criterion for determining whether any four quantities are proportional. We have only to multiply the means together, and also the extremes. If the products are equal, the quantities are proportional. If the products are not equal, the quantities are not proportional.

343. It is evident that the terms of a proportion may undergo any *change* which will not destroy the *equality* of the *ratios*; or which will leave the product of the *means* equal to the product of the *extremes*. These changes are *numerous*, but they may be reduced to a few general principles.

CASE I. *Changes in the order of the terms.*

344. If four quantities are proportional, *the order of the means, or of the extremes, or of the terms of both couplets, may be inverted without destroying the proportion.*

Thus if $a : b :: c : d$, and $12 : 8 :: 6 : 4$, then,

1. *Inverting the means,** $\left\{ \begin{array}{c} a : c :: b : d \\ 12 : 6 :: 8 : 4 \end{array} \right\}$ the 1st is to the 3d as the 2d to the 4th.

QUEST.—What rule is founded on this principle? How can you tell whether four quantities are proportional? What alterations can be made in the terms of a proportion? When the means are inverted, what is it called? When the terms of each couplet are inverted, what? If the terms of only one couplet are inverted, what is the effect on the proportion?

* This is called *alternation.* (Euclid 16. 5.)

2. *Inverting the extremes*, $\left\{\begin{array}{l} d:b::c:a \\ 4:8::6:12 \end{array}\right\}$ the 4th is to the 2d as the 3d to the 1st.

3. *Inverting the terms of each couplet*,* $\left\{\begin{array}{l} b:\ a::d:c \\ 8:12::4:6 \end{array}\right\}$ the 2d is to the 1st as the 4th to the 3d.

4. We may change the *order* of the *two couplets*. (Art. 319.)

Cor. The order of the *whole proportion* may be inverted.

N. B. If the terms of only *one* of the couplets are inverted, the proportion becomes *reciprocal*. (Art. 321.)

If $a:b::c:d$, then a is to b reeiprocally, as d to c.

Case II. *Multiplying or dividing by the same quantity.*

345. If four quantities are proportional, *the two analogous or two homologous terms may be multiplied or divided by the same quantity, without destroying the proportion.* Thus,

If $a:b::c:d$, then, if *analogous* terms are multiplied, or divided, the ratios will not be altered. (Art. 311.)

1. $ma:mb::c:d$.
2. $a:b::mc:md$.
3. $\frac{a}{m}:\frac{b}{m}::c:d$.
4. $a:b::\frac{c}{m}:\frac{d}{m}$.

If *homologous* terms be multiplied or divided, both ratios will be equally increased or diminished.

5. $ma:b::mc:d$.
6. $a:mb::c:md$.
7. $\frac{a}{m}:b::\frac{c}{m}:d$.
8. $a:\frac{b}{m}::c:\frac{d}{m}$.

Cor. All the terms may be multiplied, or divided by the same quantity. Thus $ma:mb::mc:md$, or $\frac{a}{m}:\frac{b}{m}::\frac{c}{m}:\frac{d}{m}$.

Quest.—If two analogous terms are multiplied or divided by the same quantity, what is the effect? If two homologous terms are multiplied or divided, what?

* This is technically called *inversion*.

CASE III. *Comparing one proportion with another.*

346. ***If two ratios are respectively equal to a third, they are equal to each other.*** (Euclid 11. 5.)

This is nothing more than the 7th axiom applied to ratios.

1. $\left.\begin{array}{l}\text{If } a:b::m:n \\ \text{And } c:d::m:n\end{array}\right\}$ then $a:b::c:d$, or $a:c::b:d$. (Art. 344.)

2. $\left.\begin{array}{l}\text{If } a:b::m:n \\ \text{And } m:n::c:d\end{array}\right\}$ then $a:b::c:d$, or $a:c::b:d$.

Cor. $\left.\begin{array}{l}\text{If } a:b::m:n \\ \quad m:n>c:d\end{array}\right\}$ then $a:b>c:d$. (Euclid 13. 5.)

For if the ratio of $m:n$ is greater than that of $c:d$, it is manifest that the ratio of $a:b$, which is *equal* to that of $m:n$, is also greater than that of $c:d$.

N. B. In these instances, the terms which are alike in the two proportions are the *first* two and the *last* two, and the resulting proportion is uniformly direct. But this arrangement is not essential. The order of the terms may be changed, in various ways, without affecting the equality of the ratios. (Art. 344.)

The proposition to which these instances of equality belong, is usually cited by the words, "*ex æquo*," or "*ex æquali.*" (Euclid 22. 5.)

347. Any number of proportions may be compared, in the same manner, if the first two or the last two terms in each preceding proportion, are the same with the first two or the last two in the following one.

$$\left.\begin{array}{ll}\text{Thus if} & a:b::c:d \\ \text{And} & c:d::h:l \\ \text{And} & h:l::m:n \\ \text{And} & m:n::x:y\end{array}\right\} \text{then } a:b::x:y.$$

QUEST.—When two ratios are each equal to a third, how are they to each other? How is this proposition cited in geometry? When may any number of proportions be compared in this manner?

That is, the first two terms of the first proportion have the same ratio as the last two terms of the last proportion. For it is manifest that the ratio of *all* the couplets is the same.

348. But if the two means, or the two extremes, in one proportion, be the same with the means, or the extremes, in another, the four remaining terms will be *reciprocally proportioned.*

$$\left.\begin{array}{ll}\text{If} & a:m::n:b \\ \text{And} & c:m::n:d\end{array}\right\} \text{ then } a:c::\frac{1}{b}:\frac{1}{d}, \text{ or } a:c::d:b.$$

$$\left.\begin{array}{ll}\text{For} & ab=mn \\ \text{And} & cd=mn\end{array}\right\} \text{(Art. 337.)} \quad \text{Therefore } ab=cd, \text{ and } a:c::d:b.$$

In this example, the two means in one proportion are like those in the other. But the principle will be the same, if the *extremes* are alike, or if the extremes in one proportion are like the means in the other.

$$\left.\begin{array}{ll}\text{If} & m:a::b:n \\ \text{And} & m:c::d:n\end{array}\right\} \text{then } a:c::d:b.$$

$$\left.\begin{array}{ll}\text{Or if} & a:m::n:b \\ \text{And} & m:c::d:n\end{array}\right\} \text{then } a:c::d:b.$$

The proposition in geometry which applies to this case, is usually cited by the words "*ex æquo perturbate.*" (Euc. 23. 5.)

CASE IV. *Addition and Subtraction of equal ratios.*

349. *If to or from two analogous or two homologous terms of a proportion, two other quantities having the same ratio be added or subtracted, the proportion will be preserved.* (Euclid 2. 5.)

QUEST.—If the two means or two extremes in one proportion be the same as the means or extremes in another, how are the remaining terms? How is this proposition cited in geometry? When two analogous or homologous terms are added to or subtracted from two other quantities having the same ratio, how is the proportion?

For a ratio is not altered, by adding to it, or subtracting from it, the terms of another *equal* ratio. (Art. 312.)

If $a:b::c:d$, and $a:b::m:n$,

Then by adding to, or subtracting from a and b, the terms of the equal ratio $m:n$, we have,

$$a+m:b+n::c:d, \quad \text{and} \quad a-m:b-n::c:d.$$

And by adding and subtracting m and n, to and from c and d, we have,

$$a:b::c+m:d+n, \quad \text{and} \quad a:b::c-m:d-n.$$

Here the addition and subtraction are to and from *analogous* terms. But by alternation, (Art. 344,) these terms will become *homologous*, and we shall have,

$$a+m:c::b+n:d, \quad \text{and} \quad a-m:c::b-n:d.$$

Cor. 1. This addition may evidently be extended to *any number* of equal ratios. (Euclid 2. 5. cor.)

Thus if $a:b::\left\{\begin{array}{l} c:d \\ h:l \\ m:n \\ x:y \end{array}\right\}$ then $a:b::c+h+m+x:d+l+n+y$.

Cor. 2. If $\left.\begin{array}{r} a:b::c:d \\ \text{And } m:b::n:d \end{array}\right\}$ then $a+m:b::c+n:d$. (Eu. 24. 5.)

For by alternation $\left.\begin{array}{l} a:c::b:d \\ m:n::b:d \end{array}\right\}$ then $\left\{\begin{array}{l} a+m:c+n::b:d \\ \text{or } a+m:b::c+n:d. \end{array}\right.$
And

350. *Hence, if two analogous or homologous terms be added to, or subtracted from the two others, the proportion will be preserved.*

Thus, if $a:b::c:d$, and $12:4::6:2$, then,

1. *Adding* the *last* two terms, to the *first* two.

$a+c:b+d::a:b$	$12+6:\ 4+2::12:4$
and $a+c:b+d::c:d$	$12+6:\ 4+2::\ 6:2$
or $a+c:a::b+d:b$	$12+6:12::4+\ 2:4$
and $a+c:c::b+d:d$	$12+6:\ 6::4+\ 2:2.$

2. *Adding* the two *antecedents* to the two *consequents.*

$a+b:b::c+d:d$ $\qquad$ $12+4:4::6+2:2$

$a+b:a::c+d:c$, &c. $\qquad$ $12+4:12::6+2:6$, &c.

This is called *composition.* (Euclid 18. 5.)

3. *Subtracting* the *first* two terms from the *last* two.

$c-a:a::d-b:b$, or $c-a:c::d-b:d$, &c.

4. *Subtracting* the *last* two terms from the *first* two.

$a-c:b-d::a:b$, or $a-c:b-d::c:d$, &c.

5. *Subtracting* the *consequents* from the *antecedents.*

$a-b:b::c-d:d$, or $a:a-b::c:c-d$, &c.

The alteration expressed by the last of these forms is called *conversion.*

6. *Subtracting* the *antecedents* from the *consequents.*

$b-a:a::d-c:c$, or $b:b-a::d:d-c$, &c.

7. Adding and subtracting, $a+b:a-b::c+d:c-d$.

That is, the sum of the first two terms, is to their difference, as the sum of the last two, to their difference.

Cor. If any compound quantities, arranged as in the preceding examples, are proportional, the simple quantities of which they are compounded are proportional also.

Thus, if $a+b:b::c+d:d$, then $a:b::c:d$. This is called *division.* (Euclid 17. 5.)

Case V. *Compounding Proportions.*

351. *If the corresponding terms of two or more ranks of proportional quantities be multiplied together, the products will be proportional.*

Quest.—What is composition? Conversion? Division? If the corresponding terms of two or more ranks of proportionals are multiplied together, how will the product be?

This process is called *compounding proportions.* It is the same as *compounding ratios.* It should be distinguished from what is called *composition*, which is an *addition* of the terms of a ratio. (Art. 350, 2.)

If	$a:b::c:d$	$12:4::6:2$
And	$h:l::m:n$	$10:5::8:4$
Then	$ah:bl::cm:dn$	$120:20::48:8$

For from the nature of proportion, the two ratios in the first rank are equal, and also the ratios in the second rank. And multiplying the corresponding terms is multiplying the *ratios*, (Art. 311,) that is, multiplying *equals by equals*, (Ax. 3;) so that the ratios will still be equal, and therefore the four products must be proportional.

The same proof is applicable to any number of proportions.

$$\text{If}\quad \left\{\begin{array}{l} a:b::c:d \\ h:l::m:n \\ p:q::x:y \end{array}\right\} \text{ then } ahp:blq::cmx:dny.$$

From this it is evident that if the terms of a proportion be multiplied, each into *itself*, that is, if they be *raised to any power*, they will still be proportional. (Art. 308.)

If	$a:b::c:d$	$2:4::6:12$
	$a:b::c:d$	$2:4::6:12$
Then	$a^2:b^2::c^2:d^2$	$4:16::36:144$

Proportionals will also be obtained, by *reversing* this process, that is, by extracting the *roots* of the terms.

If $a:b::c:d$, then $\sqrt{a}:\sqrt{b}::\sqrt{c}:\sqrt{d}$.

For taking the products of the extremes and means, $ad=bc$.

And extracting the root of both sides, $\sqrt{ad}=\sqrt{bc}$

That is, (Arts. 210.*a*, 339,) $\sqrt{a}:\sqrt{b}::\sqrt{c}:\sqrt{d}$.

QUEST.—What is meant by compounding proportions? What is the difference between compounding proportions and composition? If several quantities are proportional, how are like powers or roots of them?

Case VI. *Involution and Evolution of the terms.*

352. If several quantities are proportional, *their like powers or like roots are proportional.*

If $a:b::c:d$,

Then $a^n:b^n::c^n:d^n$, and $\sqrt[n]{a}:\sqrt[n]{b}::\sqrt[n]{c}:\sqrt[n]{d}$.

And $\sqrt[n]{a^m}:\sqrt[n]{b^m}::\sqrt[n]{c^m}:\sqrt[n]{d^m}$, that is, $a^{\frac{m}{n}}:b^{\frac{m}{n}}::c^{\frac{m}{n}}:d^{\frac{m}{n}}$.

353. If the terms in one rank of proportionals be *divided* by the corresponding terms in another rank, the quotients will be proportional.

This is sometimes called the *resolution* of ratios.

If	$a:b::c:d$	$12:6::18:9$
And	$h:l::m:n$	$6:2::\ 9:3$
Then	$\frac{a}{h}:\frac{b}{l}::\frac{c}{m}:\frac{d}{n}$	$\frac{12}{6}:\frac{6}{2}::\frac{18}{9}:\frac{9}{3}$

This is merely *reversing* the process in Art. 351, and may be demonstrated in a similar manner.

N. B. This should be distinguished from what geometers call *division*, which is a *subtraction* of the terms of a ratio. (Art. 350, 7.)

354. When proportions are compounded by multiplication, it will often be the case that the *same factor* will be found in two analogous or two homologous terms.

Thus if $a:b::c:d$

And $m:a::n:c$

$$am:ab::cn:cd.$$

Here a is in the first two terms, and c in the last two. Dividing by these, (Art. 345,) the proportion becomes

$m:b::n:d$. Hence,

Quest.—What is meant by the resolution of ratios?

355. *In compounding proportions, equal factors or divisors* in two analogous or homologous terms, may be *rejected.*

$$\text{If}\left\{\begin{array}{l} a:b::c:d \\ b:h::d:l \\ h:m::l:n \end{array}\right. \qquad \begin{array}{c} 12:4::9:3 \\ 4:8::3:6 \\ 8:20::6:15 \end{array}$$

$$\text{Then} \quad \overline{a:m::c:n} \qquad \overline{12:30::9:15}$$

This rule may be applied to the cases, to which the terms "*ex æquo*" and "*ex æquo perturbate*" refer. (Arts. 346 *a*, 348.) One of the methods may serve to verify the other.

356. When four quantities are proportional, if the *first* be greater than the *second*, the *third* will be greater than the *fourth*; if equal, equal; if less, less.

$$\text{Suppose } a:b::c:d\text{; then if } \left\{\begin{array}{l} a=b,\ c=d. \\ a>b,\ c>d. \\ a<b,\ c<d. \end{array}\right.$$

357. If four quantities are proportional, their *reciprocals* are proportional; and v. v.

$$\text{If } a:b::c:d\text{, then } \frac{1}{a}:\frac{1}{b}::\frac{1}{c}:\frac{1}{d}.$$

For in each of these proportions, we have, by reduction, $ad=bc$.

PROBLEMS IN GEOMETRICAL PROPORTION.

Prob. 1. Divide the number 49 into two such parts, that the greater increased by 6, may be to the less diminished by 11 as 9 to 2.

QUEST.—In compounding proportions, what may be done with equal factors or divisors? When four quantities are proportional, if the first is greater than the second, how is the third? If equal? If less? If four quantities are proportional, how are their reciprocals?

Let $x=$ the greater, and $49-x=$ the less.

By the conditions proposed, $x+6:38-x::9:2$

Adding terms, (Art. 350, 2,) $x+6:44::9:11$

Dividing the consequents, (Art. 345, 8,) $x+6:4::9:1$

Multiplying the extremes and means, $x+6=36$. And $x=30$.

Prob. 2. What number is that, to which if 1, 5 and 13, be severally added, the first sum shall be to the second, as the second to the third? 3

Prob. 3. Find two numbers, the greater of which shall be to the less, as their sum to 42; and as their difference to 6.

Prob. 4. Divide the number 18 into two such parts, that the squares of those parts may be in the ratio of 25 to 16. 10 & 8

Prob. 5. Divide the number 14 into two such parts, that the quotient of the greater divided by the less, shall be to the quotient of the less divided by the greater as 16 to 9. 8, 6

Prob. 6. If the number 20 be divided into two parts, which are to each other in the *duplicate* ratio of 3 to 1, what number is a mean proportional between those parts? 6

Prob. 7. There are two numbers whose product is 24, and the difference of their cubes, is to the cube of their difference as 19 to 1. What are the numbers? 6, 4

Prob. 8. There are two numbers in the proportion of 5:6; the first being increased by 4 and the last by 6, the proportion will be as 4:5. What are the numbers? 20 & 24

Prob. 9. A farmer has a quantity of corn in his granary, and sells a certain number of bushels, which is to the number of bushels remaining as 4:5. He then feeds out 15 bushels, which is to the number sold as 1:2. How many bushels had he at first, and how many did he sell?

Prob. 10. There are two numbers whose product is 135, and the difference of their squares, is to the square of their difference as 4 to 1. What are the numbers? 15, 9.

Prob. 11. What two numbers are those, whose difference, sum, and product, are as the numbers 2, 3 and 5, respectively?

16 Prob. 12. Divide the number 24 into two such parts, that their product shall be to the sum of their squares as 3 to 10.

Prob. 13. In a mixture of rum and brandy, the difference between the quantities of each, is to the quantity of brandy as 100 is to the number of gallons of rum; and the same difference is to the quantity of rum, as 4 to the number of gallons of brandy. How many gallons are there of each? 5 — 75

Prob. 14. There are two numbers which are to each other as 3 to 2. If 6 be added to the greater and subtracted from the less, the sum and remainder will be to each other as 3 to 1. What are the numbers? 24 — 16

Prob. 15. There are two numbers whose product is 320; and the difference of their cubes, is to the cube of their difference as 61 to 1. What are the numbers? 16 ; 20

Prob. 16. There are two numbers, which are to each other, in the duplicate ratio of 4 to 3; and 24 is a mean proportional between them. What are the numbers? 32 — 16

CONTINUED GEOMETRICAL PROPORTION OR PROGRESSION.

358. When all the ratios of a series of proportionals are equal, the quantities are said to be in *continued proportion* or *progression*. (Art. 322.)

As arithmetical proportion continued is arithmetical progression, so geometrical proportion continued is geometrical progression. It is sometimes called progression by quotient.

QUEST.—What is continued geometrical proportion? What else is it called?

The numbers 64, 32, 16, 8, 4, are in continued geometrical proportion. (Art. 322.)

In this series, if each preceding term be *divided* by the common ratio, the quotient will be the following term. Thus,

$$\tfrac{64}{2}=32, \text{ and } \tfrac{32}{2}=16, \text{ and } \tfrac{16}{2}=8, \text{ and } \tfrac{8}{2}=4.$$

If the order of the series be *inverted*, the proportion will still be preserved, (Art. 344;) and the common divisor will become a multiplier. In the series 4, 8, 16, 32, 64, &c

$$4\times2=8, \text{ and } 8\times2=16, \text{ and } 16\times2=32, \text{ \&c.}$$

359. *Quantities then are in geometrical progression, when they increase by a common multiplier, or decrease by a common divisor.*

This common multiplier or divisor is called the *ratio*. For most purposes, however, it will be more simple to consider the ratio as always a *multiplier*, either integral or fractional.

In the series 64, 32, 16, 8, 4, the ratio is either 2 considered as a divisor, or $\frac{1}{2}$ considered as a multiplier.

360. When several quantities are in *continued* proportion, the *number of couplets*, and of course the *number of ratios*, is *one less* than the number of quantities. Thus the five proportional quantities a, b, c, d, e, form four couplets containing four ratios; and the ratio of $a:e$ is equal to the ratio of $a^4:b^4$, that is, the ratio of the fourth power of the first quantity, to the fourth power of the second. Hence,

361. If *three* quantities are proportional, *the first is to the third, as the square of the first to the square of the second; or as the square of the second, to the square of the third.* In other words, the first has to the third, a *duplicate* ratio of the first to the second. And conversely, if the first of the three

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QUEST.—When are quantities said to be in geometrical progression? What is the common multiplier or divisor called? In a series of continued proportion, how many couplets and ratios are there? When there are three proportionals what ratio has the first to the third?

quantities is to the third, as the square of the first to the square of the second, the three quantities are proportional.

If $a:b::b:c$, then $a:c::a^2:b^2$. And universally,

362. If *several* quantities are in continued proportion, the ratio of the *first* to the *last* is equal to one of the intervening ratios raised to a power whose index is one less than the number of quantities.

If there are *four* proportionals a, b, c, d, then $a:d::a^3:b^3$.
If there are *five* a, b, c, d, e; $a:e::a^4:b^4$, &c.

363. If several quantities are in continued proportion, they will be proportional when the order of the whole is *inverted*. This has already been proved with respect to *four* proportional quantities. (Art. 344, cor.) It may be extended to any number of quantities.

Between the numbers,	64, 32, 16, 8, 4,
The ratios are,	2, 2, 2, 2,
Between the same inverted,	4, 8, 16, 32, 64,
The ratios are,	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$.

So if the order of any proportional quantities be inverted, the ratios in one series will be the *reciprocals* of those in the other. For by the inversion each antecedent becomes a consequent, and *v. v.* but the ratio of a consequent to its antecedent is the reciprocal of the ratio of the antecedent to the consequent. (Art. 305.) That the reciprocals of equal quantities are themselves equal, is evident from Ax. 4.

364. To investigate the properties of geometrical progression, we may take nearly the same course, as in arithmetical progression, observing to substitute continual *multiplication and division*, instead of addition and subtraction. It is evident, in the first place, that,

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QUEST.—When several quantities are in continued proportion, what ratio has the first to the last? If the series is inverted, what effect has it? How are the ratios in one series, compared with those of another, when the order is inverted?

365. In an ascending geometrical series, each succeeding term is found, by *multiplying the ratio* into the preceding term.

If the first term is a, and the ratio r,

Then $a+r=ar$, the second term, $ar\times r=ar^2$, the third,

$ar^2+r=ar^3$, the fourth, $ar^3\times r=ar^4$, the fifth, &c.

And the series is a, ar, ar^2, ar^3, ar^4, ar^5, &c.

366. If the *first term* and the *ratio* are the *same*, the progression is simply a series of powers.

If the first term and ratio are each equal to r,

Then $r\times r=r^2$, the second term, $r^2\times r=r^3$, the third,

$r^3\times r=r^4$, the fourth, $r^4\times r=r^5$, the fifth.

And the series is r, r^2, r^3, r^4, r^5, r^6, &c.

367. In a *descending* series, each succeeding term is found by *dividing* the preceding term by the ratio, or *multiplying* by the fractional ratio.

If the first term is ar^6, and the ratio r,

The second term is $\frac{ar^6}{r}$, or $ar^6\times\frac{1}{r}=ar^5$,

And the series is ar^6, ar^5, ar^4, ar^3, ar^2, ar, a, &c.

If the first term is a, and the ratio r,

The series is a, $\frac{a}{r}$, $\frac{a}{r^2}$, $\frac{a}{r^3}$, &c., or a, ar^{-1}, ar^{-2}, &c.

368. By attending to the series $\overset{\text{1st}}{a}$, $\overset{\text{2d}}{ar}$, $\overset{\text{3d}}{ar^2}$, $\overset{\text{4th}}{ar^3}$, $\overset{\text{5th}}{ar^4}$, $\overset{\text{6th}}{ar^5}$, &c., it will be seen that, in each term, the exponent of the power of the ratio, is *one less* than the number of the term.

If then $a=$ the first term, $r=$ the ratio,

$z=$ the last, $n=$ the number of terms;

we have the equation $z=ar^{n-1}$, the last term, that is,

QUEST.—In an ascending geometrical series how is each succeeding term found? When the first term and ratios are the same, what is the progression? How is each term found in a descending series?

369. In geometrical progression, *the last term is equal to the product of the first into that power of the ratio whose index is one less than the number of terms.*

When the last term and the ratio are the *same*, the equation becomes $z=rr^{n-1}=r^n$. (Art 366.)

370. Of the four quantities a, z, r and n, any *three* being given, the other may be found.

1. By the last article,

$$z=ar^{n-1}=\text{ the } \textit{last term}.$$

2. Dividing by r^{n-1},

$$\frac{z}{r^{n-1}}=a=\text{ the } \textit{first term}.$$

3. Dividing the 1st by a, and extracting the root,

$$\left(\frac{z}{a}\right)^{\frac{1}{n-1}}=r=\text{ the } \textit{ratio}.$$

371. By the last equation may be found any number of *geometrical means*, between two given numbers. If $m=$ the number of means, $m+2=n$, the *whole* number of terms. Substituting $m+2$, for n, in the equation, we have

$$\left(\frac{z}{a}\right)^{\frac{1}{m+1}}=r, \text{ the ratio.}$$

When the ratio is found, the means are obtained by continued multiplication.

Prob. 1. Find two geometrical means between 4 and 256. Ans. The ratio is 4, and the series is 4, 16, 64, 256.

Prob. 2. Find three geometrical means between $\frac{1}{9}$ and 9.

372. The next thing to be attended to, is the rule for finding the *sum of all the terms.*

QUEST.—What is the last term equal to? What is the first term equal to? How find the ratio?

If any term, in a geometrical series, be multiplied by the ratio, the product will be the succeeding term. (Art. 365.) Of course, if *each* of the terms be multiplied by the ratio, a new series will be produced, in which all the terms except the last will be the same, as all except the first in the other series. To make this plain, let the new series be written under the other, in such a manner, that each term shall be removed one step to the right of that from which it is produced in the line above.

Take, for instance, the series, 2, 4, 8, 16, 32
Multiplying each term by the ratio, 4, 8, 16, 32, 64

Here it will be seen at once, that the last four terms in the upper line are the same as the first four in the lower line. The only terms which are not in *both*, are the *first* of the one series, and the *last* of the other. So that when we subtract the one series from the other, all the terms except these two will disappear, by balancing each other.

If the given series is, $a, ar, ar^2, ar^3, \ldots . ar^{n-1}$.
Then mult. by r, we have $ar, ar^2, ar^3, \ldots . ar^{n-1}, ar^n$.

Now let $s=$ the sum of the terms.

Then, $s=a+ar+ar^2+ar^3, \ldots . +ar^{n-1}$,
And mult. by r, $rs= \quad ar+ar^2+ar^3, \ldots . +ar^{n-1}+ar^n$.

Subt. the first equation from the second, $rs-s=ar^n-a$
And dividing by $(r-1,)$ (Art. 98,) $s=\dfrac{ar^n-a}{r-1}$.

In this equation, ar^n is the last term in the new series, and is therefore the product of the ratio into the last term in the *given series*.

Therefore, $s=\dfrac{rz-a}{r-1}$, that is,

373. To find the sum of a geometrical series.

Multiply the last term into the ratio, from the product subtract the first term, and divide the remainder by the ratio less one.

Obser. From the above formula, in connexion with the one in Art. 368, there may be the same variety of other formulas deduced as in Art. 333. The others, however, involve principles with which, it is presumed, the pupil is not yet acquainted.

Prob 3. If in a series of numbers in geometrical progression, the first term is 6, the last term 1458, and the ratio 3, what is the sum of all the terms?

$$\text{Ans. } s = \frac{rz-a}{r-1} = \frac{3 \times 1458 - 6}{3-1} = 2184.$$

Prob. 4. If the first term of a decreasing geometrical series is $\frac{1}{2}$, the ratio $\frac{1}{3}$, and the number of terms 5, what is the sum of the series?

Prob. 5. What is the sum of the series, 1, 3, 9, 27, &c., to 12 terms?

Prob. 6. What is the sum of ten terms of the series 1, $\frac{2}{3}$, $\frac{4}{9}$, $\frac{8}{27}$, &c.?

Prob. 7. If the first term of a series is 2, the ratio 2, and the number of terms 13, what is the last term?

Prob. 8. What is the 12th term of a series, the first term of which is 3, and the ratio 3?

Prob. 9. A man bought a horse, agreeing to give 1 cent for the first nail in his shoes, three for the second, and so on. The shoes contained 32 nails; what was the cost of the horse?

374. *Quantities in geometrical progression are proportional to their differences.*

Quest.—How is the sum of a geometrical series found? If quantities are in geometrical progression, how are they to their differences? How are their differences to each other?

Let the series be a, ar, ar^2, ar^3, ar^4, &c.

By the nature of geometrical progression,

$$a : ar :: ar : ar^2 :: ar^2 : ar^3 :: ar^3 : ar^4, \text{ \&c.}$$

In each couplet let the antecedent be subtracted from the consequent, according to Art. 350, 6.

Then $a : ar :: ar-a : ar^2-ar :: ar^2-ar : ar^3-ar^2$, &c.

That is, the first term is to the second, as the difference between the first and second, to the difference between the second and third; and as the difference between the second and third, to the difference between the third and fourth, &c.

Cor. If quantities are in geometrical progression, their *differences* are also in geometrical progression.

Thus the numbers 3, 9, 27, 81, 243, &c.

And their differences 6, 18, 54, 162, &c., are in geometrical progression.

375. Problems in geometrical progression may be solved, as in other parts of algebra, by means of equations.

Prob. 10. Find three numbers in geometrical progression, such that their sum shall be 14, and the sum of their squares 84.

Let the three numbers be x, y and z.

By the conditions, $x : y :: y : z$, or $xz = y^2$

And $x + y + z = 14$

And $x^2 + y^2 + z^2 = 84$

Ans. 2, 4 and 8.

Prob. 11. There are three numbers in geometrical progression whose product is 64, and the sum of their cubes is 584. What are the numbers? 2 – 4

Prob. 12. There are three numbers in geometrical progression: the sum of the first and last is 52, and the square of the mean is 100. What are the numbers? 2 –

Prob. 13. Of four numbers in geometrical progression, the sum of the first two is 15; and the sum of the last two is 60. What are the numbers?

Prob. 14. A gentleman divided 210 dollars among three servants, in such a manner that their portions were in geometrical progression; and the first had 90 dollars more than the last. How much had each?

Prob. 15. There are three numbers in geometrical progression, the greatest of which exceeds the least by 15; and the difference of the squares of the greatest and the least, is to the sum of the squares of all the three numbers as 5 to 7. What are the numbers?

Prob. 16. There are four numbers in geometrical progression, the second of which is less than the fourth by 24; and the sum of the extremes is to the sum of the means as 7 to 3. What are the numbers?

SECTION XV.

EVOLUTION OF COMPOUND QUANTITIES.

376. Rule.—I. *Arrange the terms according to the powers of one of the letters, so that the highest power shall stand first, the next highest next, &c.*

II. *Take the root of the first term, for the first term of the required root:*

III. *Subtract the power from the given quantity, and divide the first term of the remainder by the first term of the root in*

Quest.—How should the terms be arranged to extract the root of a compound quantity? What are the other steps?

volved to the next inferior power and multiplied by the index of the given power; the quotient will be the next term of the root.*

IV. *Subtract the power of the terms already found from the given quantity, and using the same divisor, proceed as before.*

PROOF. *This rule verifies itself.* For the root, whenever a new term is added to it, is involved, for the purpose of subtracting its power from the given quantity: and when the power is *equal* to this quantity, it is evident the true root is found.

1. Extract the cube root of

$a^6+3a^5-3a^4-11a^3+6a^2+12a-8(a^2+a-2.$

a^6, the first subtrahend.

$3a^4)^*\ 3a^5$, &c., the first remainder.

$a^6+3a^5+3a^4+a^3$, the 2d subtrahend.

$3a^4)^*\ {}^*\ -6a^4$, &c., the 2d remainder.

$a^6+3a^5-3a^4-11a^3+6a^2+12a-8.$

2. Find the 4th root of $a^4+8a^3+24a^2+32a+16$. a + 2
3. Find the 5th root of $a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$. a + b
4. Find the cube root of $a^3-6a^2b+12ab^2-8b^3$. a − 2b
5. Find the 2d root of $4a^2-12ab+9b^2+16ah-24bh+16h^2$. 2a − 3b + 4h

N. B. In finding the divisor in the 5th example, the term $2a$ in the root is not involved, because the power next below the square is the first power.

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* By the *given power* is meant a power of the same name with the required root. As powers and roots are correlative terms, any quantity is the square of its square root, the cube of its cube root, &c.

377. The *square* root may be extracted by the following

RULE. I. *Arrange the terms of the given quantity according to the powers of one of the letters, take the root of the first term, for the first term of the required root, and subtract the power from the given quantity.*

II. *Bring down two other terms for a dividend. Divide by double the root already found, and add the quotient both to the root, and to the divisor. Multiply the divisor thus increased, into the term last placed in the root, and subtract the product from the dividend.*

III. *Bring down two or three additional terms and proceed as before.*

PROOF. *Multiply the root into itself, and if the product is equal to the given quantity, the work is right.*

6. What is the square root of

$a^2+2ab+b^2+2ac+2bc+c^2(a+b+c$

a^2, the first subtrahend.

$2a+b)$ * $2ab+b^2$

Into $b=$ $2ab+b^2$, the second subtrahend.

$2a+2b+c)$ * * $2ac+2bc+c^2$

Into $c=$ $2ac+2bc+c^2$, the third subtrahend.

Proof. The square of the root $a+b+c$, is equal to the given quantity.

For $(a+b)^2=a^2+2ab+b^2=a^2+(2a+b)\times b$. (Art. 97.)

And substituting $h=a+b$, the square $h^2=a^2+(2a+b)\times b$.

And $(a+b+c)^2=(h+c)^2=h^2+(2h+c)\times c$;

that is, restoring the values of h and h^2,

$$(a+b+c)^2=a^2+(2a+b)\times b+(2a+2b+c)\times c.$$

In the same manner, it may be proved, that, if another term be added to the root, the power will be increased, by the product of that term into itself, and into twice the sum of the preceding terms.

QUEST.—What is the rule for extracting the square root?

The demonstration will be substantially the same, if some of the terms be *negative*.

7. Find the square root of $1-4b+4b^2+2y-4by+y^2$.

8. Find the square root of $a^6-2a^5+3a^4-2a^3+a^2$. a^3-a^2+a

9. Find the square root of $a^4+4a^2b+4b^2-4a^2-8b+4$. a^2+2b-2

378. It will frequently facilitate the extraction of roots, to consider the index as composed of two or more *factors*.

Thus $a^{\frac{1}{4}}=a^{\frac{1}{2}\times\frac{1}{2}}$. And $a^{\frac{1}{6}}=a^{\frac{1}{3}\times\frac{1}{2}}$. That is,

The *fourth* root is equal to the *square* root of the *square* root;

The *sixth* root is equal to the *square* root of the *cube* root;

The *eighth* root is equal to the *square* root of the *fourth* root, &c.

To find the *sixth* root, therefore, we may first extract the *cube* root, and then the *square* root of this.

10. Find the square root of $x^4-4x^3+6x^2-4x+1$. x^2-2x+1

11. Find the cube root of $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$.

12. Find the square root of $4x^4-4x^3+13x^2-6x+9$. $2x^2-x+3$

13. Find the 4th root of $16a^4-96a^3x+216a^2x^2-216ax^3+81x^4$.

14. Find the 5th root of $x^5+5x^4+10x^3+10x^2+5x+1$. $x+1$

15. Find the 6th root of $a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6$.

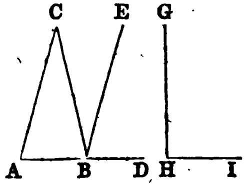
QUEST.—How may the extraction of roots be facilitated? What is the fourth root equal to? The sixth? The eighth? How then may we find the sixth root?

SECTION XVI.

APPLICATION OF ALGEBRA TO GEOMETRY.*

Art. 379. It is often expedient to make use of the algebraic notation, for expressing the relations of geometrical quantities, and to throw the several steps of a demonstration into the form of equations. By this, the nature of the reasoning is not altered. It is only translated into a different *language.* *Signs* are substituted for *words*, but they are intended to convey the same meaning. A great part of the demonstrations in geometry really consist of a series of equations, though they may not be presented to us under the algebraic forms. Thus the proposition, that *the sum of the three angles of a triangle is equal to two right angles*, (Euc. 32. 1,) may be demonstrated, either in common language, or by means of the signs used in algebra.

Fig. 1.



Let the side AB, of the triangle ABC, (Fig. 1,) be continued to D; let the line BE be parallel to AC; and let GHI be a right angle.

The demonstration, in words, is as follows:

1. The angle EBD is *equal* to the angle BAC, (Euc. 29. 1.)
2. The angle CBE is *equal* to the angle ACB.
3. Therefore, the angle EBD *added* to CBE, that is, the angle CBD, is *equal* to BAC *added* to ACB.
4. If to these equals, we add the angle ABC, the angle CBD *added* to ABC, is *equal* to BAC *added* to ACB and ABC.

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* This section is to be read *after* the Elements of Geometry.

5. But CBD *added* to ABC, is *equal* to twice GHI, that is, to two right angles. (Euc. 13. 1.)
6. Therefore, the angles BAC, and ACB, and ABC, are together equal to twice GHI, or two right angles.

Now by substituting the sign +, for the word *added*, or *and*, and the sign =, for the word *equal*, we shall have the same demonstration in the following form.

1. By Euclid 29. 1. $EBD=BAC$
2. And $CBE=ACB$
3. Add the two equations $EBD+CBE=BAC+ACB$
4. Add ABC to both sides $CBD+ABC=BAC+ACB+ABC$
5. But by Euclid 13. 1. $CBD+ABC=2GHI$
6. Make the 4th and 5th equal $BAC+ACB+ABC=2GHI$.

By comparing, one by one, the steps of these two demonstrations, it will be seen that they are precisely the same, except that they are differently expressed.

380. It will be observed that the notation in the example just given, differs, in one respect, from that which is generally used in algebra. Each quantity is represented, not by a *single letter*, but by *several*. In common algebra when one letter stands immediately before another as ab, without any character between them, they are to be considered as *multiplied* together.

But in geometry, AB is an expression for a *single line*, and not for the product of A into B. Multiplication is denoted, either by a point or by the sign $\times$. The product of AB into CD, is $AB \, . \, CD$, or $AB \times CD$.

381. There is no impropriety, however, in representing a geometrical quantity by a single letter. We may make b stand for a line or an angle, as well as for a number.

If, in the example above, we put the angle

EBD$=a$,	ACB$=d$,	ABC$=h$,
BAC$=b$,	CBD$=g$,	GHI$=l$,
CBE$=c$,		

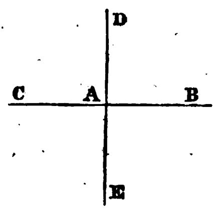
the demonstration will stand thus:

1. By Euclid 29. 1,	$a=b$
2. And	$c=d$
3. Adding the two equations,	$a+c=b+d$
4. Adding h to both sides,	$g+h=b+d+h$
5. By Euclid, 13, 1,	$g+h=2l$
6. Making the 4th and 5th equal,	$b+d+h=2l$.

This notation is apparently more simple than the other; but it deprives us of what is of great importance in geometrical demonstrations, continual and easy reference to the figure. To distinguish the two methods, *capitals* are generally used for that which is peculiar to geometry; and *small letters*, for that which is properly algebraic.

382. If a line whose length is measured from a given point or line, be considered *positive*; a line proceeding in the *opposite* direction must be considered *negative*. If AB (Fig. 2,) reckoned from DE on the *right*, is positive; AC on the *left* is negative.

Fig. 2.



Hence, if, in the course of a calculation, the algebraic value of a line is found to be *negative*; it must be measured in a direction opposite to that which, in the same process, has been considered positive. (Art. 162.*a.*)

383. In algebraic calculations, there is frequent occasion for *multiplication*, *division*, *involution*, &c. But how, it may

be asked, can *geometrical* quantities be multiplied into each other? One of the factors, in multiplication, is always to be considered as a *number*. The operation consists in repeating the multiplicand as many times as there are *units* in the multiplier. How then can a *line*, a *surface*, or a *solid*, become a multiplier?

To explain this it will be necessary to observe, that whenever one geometrical quantity is multiplied into another, some *particular extent* is to be considered *the unit*. It is immaterial what this extent is, provided it remains the same in different parts of the same calculation. It may be an inch, a foot, a rod, or a mile. If, for instance, one of the lines be a foot long, and the other half a foot; the factors will be, one 12 inches, and the other 6, and the product will be 72 inches. Though it would be absurd to say that one line is to be repeated *as often as another is long*; yet there is no impropriety in saying, that one is to be repeated as many times as there are feet or rods in the other. This, the nature of a calculation often requires.

384. If the line which is to be the multiplier, is only a *part* of the length taken for the unit, the product is a like part of the multiplicand. (Art. 71.) Thus, if one of the factors is 6 inches, and the other half an inch, the product is 3 inches.

385. Instead of referring to the measures in common use, as inches, feet, &c., it is often convenient to fix upon one of the lines in a figure, as the unit with which to compare all the others. When there are a number of lines drawn within and about a *circle*, the *radius* is commonly taken for the unit. This is particularly the case in trigonometrical calculations.

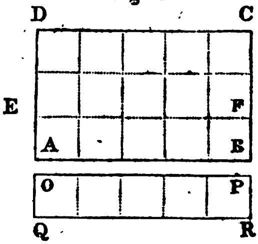
386. The observations which have been made concerning lines, may be applied to *surfaces* and *solids*. There may be occasion to multiply the *area* of a figure by the number of inches in some given line.

But here another difficulty presents itself. The product of two lines is often spoken of as being equal to a *surface;* and the product of a line and a surface, as equal to a *solid.* But if a line has no *breadth,* how can the multiplication, that is, the *repetition,* of a line produce a surface? And if a surface has no thickness, how can a repetition of it produce a solid?

387. In answering these inquiries it must be admitted, that measures of length do not belong to the same class of magnitudes with superficial or solid measures; and that none of the steps of a calculation can, properly speaking, transform the one into the other. But, though a line cannot become a surface or a solid, yet the several measuring units in common use are so adapted to each other, that squares, cubes, &c., are bounded by lines of the same name. Thus the side of a square inch is a linear inch; that of a square rod, a linear rod, &c. The *length* of a linear inch is, therefore, the same as the length or breadth of a square inch.

If then several square inches are placed together, as from Q to R, (Fig. 3,) the *number* of them in the parallelogram OR is the same as the number of linear inches in the side QR: and if we know the length of this, we have of course the area of the parallelogram, which is here supposed to be one inch wide.

Fig. 3.



But, if the breadth is *several* inches, the larger parallelogram contains as many smaller ones, each an inch wide, as there are inches in the whole breadth. Thus, if the parallelogram AC, (Fig. 3,) is 5 inches long and 3 inches broad, it may be divided into three such parallelograms as OR. To obtain, then, the number of squares in the large parallelo-

gram, we have only to multiply the number of squares in one of the small parallelograms, into the number of such parallelograms contained in the whole figure. But the number of square inches in one of the small parallelograms is equal to the number of linear inches in the *length* AB. And the number of small parallelograms is equal to the number of linear inches in the *breadth* BC. It is therefore said concisely, that *the area of a parallelogram is equal to its length multiplied into its breadth.*

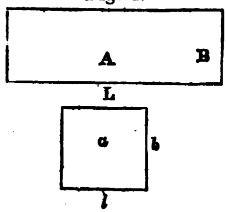
388. We hence obtain a convenient algebraic expression, for the area of a right-angled parallelogram. If two of the sides perpendicular to each other are AB and BC, the expression for the area is $AB \times BC$; that is, putting a for the area,

$$a = AB \times BC.$$

It must be remarked, however, that when AB stands for a *line*, it contains only *linear* measuring units; but when it enters into the expression for the area, it is supposed to contain *superficial* units of the same name.

389. The expression for the area may also be derived by another method more simple, but less satisfactory perhaps to some. Let a (Fig. 4,) represent a square inch, foot, rod, or other measuring unit; and let b and l be two of its sides. Also, let A be the area of any right-angled parallelogram, B its breadth, and L its length. Then it is evident, that, if the breadth of each were the same, the areas would be as the lengths; and, if the length of each were the same, the areas would be as the breadths.

Fig. 4.



That is, $A : a :: L : l$, when the breadth is given;

And $A : a :: B : b$, when the length is given;

Therefore, $A : a :: B \times L : b \times l$, when both vary.

That is, the area is as the *product* of the *length* and *breadth.*

390. Hence, in quoting the elements of geometry, the term *product* is frequently substituted for *rectangle*. And whatever is there proved concerning the equality of certain rectangles, may be applied to the product of the lines which contain the rectangles. (Legendre 166)

391. The area of an *oblique* parallelogram is also obtained by multiplying the base into the perpendicular height. Thus the expression for the area of the parallelogram ABNM (Fig. 5,) is $MN \times AD$, or $AB \times BC$. For by Art. 388, $AB \times BC$ is the area of the right angled parallelogram ABCD; and by Euclid 36. 1, parallelograms upon equal bases, and between the same parallels, are equal; that is, ABCD is equal to ABNM.

Fig. 5.

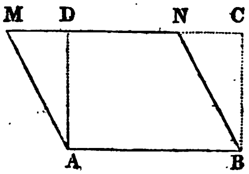
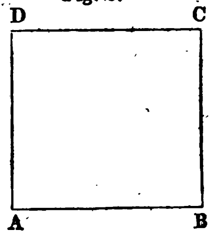


Fig. 6.



392. The area of a *square* is obtained by multiplying one of the sides *into itself*. Thus the expression for the area of the square AC, (Fig. 6,) is $(AB)^2$, that is, $a=(AB)^2$.

For the area is equal to $AB \times BC$. (Art. 388.)

But $AB=BC$, therefore, $AB \times BC = AB \times AB = (AB)^2$.

393. The area of a *triangle* is equal to *half* the product of the base and height. Thus the area of the triangle ABG, (Fig. 7,) is equal to half AB into GH or its equal BC, that is,

$$a=\tfrac{1}{2}AB \times BC.$$

For the area of the parallelogram ABCD is $AB \times BC$, (Art. 388.) And by Euclid 41. 1, if a parallelogram and a

triangle are upon the same base, and between the same parallels, the triangle is half the parallelogram. (Leg. 168.)

394. Hence, an algebraic expression may be obtained for the area of any figure whatever, which is bounded by right lines. For every such figure may be divided into triangles.

Fig. 7.

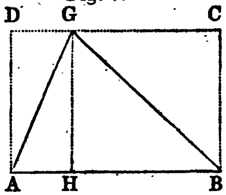
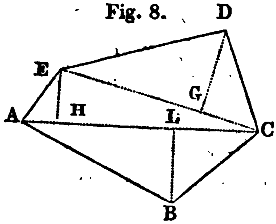


Fig. 8.



Thus the right-lined figure, ABCDE (Fig. 8,) is composed of the triangles ABC, ACE and ECD.

The area of the triangle	$ABC = \frac{1}{2}AC \times BL$,
That of the triangle	$ACE = \frac{1}{2}AC \times EH$,
That of the triangle	$ECD = \frac{1}{2}EC \times DG$.

The area of the whole figure is, therefore, equal to

$$(\tfrac{1}{2}AC \times BL) + (\tfrac{1}{2}AC \times EH) + (\tfrac{1}{2}EC \times DG).$$

395. The expression for the *superficies* has here been derived from that of a *line* or *lines*. It is frequently necessary to *reverse* this order; to find a *side* of a figure, from knowing its *area*.

If the number of square inches in the parallelogram ABCD (Fig. 3,) whose breadth BC is 3 inches, be divided by 3, the quotient will be a parallelogram, ABEF, *one* inch wide, and of the same length with the larger one. But the length of the small parallelogram is the length of its side AB. The number of *square* inches in one is the same as the number of *linear* inches in the other. (Art. 387.) If, therefore, the area of the large parallelogram be represented by a, the

side $AB=\frac{a}{BC}$, that is, *the length of a parallelogram is found by dividing the area by the breadth.*

If a be put for the area of a *square* whose side is AB,

Then by Art. 392, $a=(AB)^2$,

And extracting both sides $\sqrt{a}=AB$.

That is, *the root of the square is found, by extracting the square root of the number of measuring units in its area.*

396. If AB be the base of a *triangle* and BC its perpendicular height;

Then by Art. 393, $a=\frac{1}{2}BC\times AB$

And dividing by $\frac{1}{2}BC$, $\frac{a}{\frac{1}{2}BC}=AB$.

That is, *the base of a triangle is found, by dividing the area by half the height.*

397. As a *surface* is expressed by the product of its length and breadth; the contents of a *solid* may be expressed by the product of its length, breadth and depth. It is necessary to bear in mind, that the measuring unit of solids is a *cube;* and that the side of a cubic inch is a square inch; the side of a cubic foot, a square foot, &c.

Let ABCD (Fig. 3,) represent the base of a parallelopiped, five inches long, three inches broad, and *one* inch deep. It is evident there must be as many *cubic* inches in the solid, as there are *square* inches in its base. And, as the product of the lines AB and BC gives the area of this base, it gives, of course, the contents of the solid. But suppose that the depth of the parallelopiped, instead of being *one* inch, is *four* inches. Its contents must be four times as great. If, then, the length be AB, the breadth BC, and the depth CO, the expression for the solid contents will be, $AB\times BC\times CO$.

398. By means of the algebraic notation, a geometrical demonstration may often be rendered much more simple and

concise, than in ordinary language. The proposition, (Euc. 4. 2,) that when a straight line is divided into two parts, the square of the whole line is equal to the squares of the two parts, together with twice the product of the parts, is demonstrated, by involving a binomial.

Let the side of a square be represented by s;
And let it be divided into two parts, a and b.

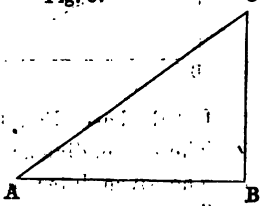
By the supposition, $s=a+b$

And squaring both sides, $s^2=a^2+2ab+b^2$.

That is, s^2 the square of the whole line, is equal to a^2 and b^2, the squares of the two parts, together with $2ab$, twice the product of the parts.

399. The algebraic notation may also be applied, with great advantage, to the solution of geometrical *problems*. In doing this, it will be necessary, in the first place, to raise an algebraic equation from the geometrical relations of the quantities given and required; and then by the usual reductions, to find the value of the unknown quantity in this equation.

Fig. 9.



Prob. 1. Given the *base*, and the *sum* of the hypothenuse and perpendicular, of the right angled triangle ABC, (Fig. 9,) to find the perpendicular.

Let the base $AB=b$

The perpendicular $BC=x$

The sum of hyp. and perp. $x+AC=a$

Then transposing x, $AC=a-x$

1. By Euclid 47. 1, $(BC)^2+(AB)^2=(AC)^2$

2. That is, by the notation, $x^2+b^2=(a-x)^2=a^2-2ax+x^2$.

And, $x=\dfrac{a^2-b^2}{2a}=BC$, the side required. Hence,

'In a right angled triangle, the perpendicular is equal to the square of the sum of the hypothenuse and perpendicular, diminished by the square of the base, and divided by twice the sum of the hypothenuse and perpendicular.'

It is applied to particular cases by substituting *numbers* for the letters a and b. Thus if the base is 8 feet, and the sum of the hypothenuse and perpendicular 16, the expression $\frac{a^2-b^2}{2a}$ becomes $\frac{16^2-8^2}{2\times 16}=6$, the perpendicular; and this subtracted from 16, the sum of the hypothenuse and perpendicular, leaves 10, the length of the hypothenuse.

Prob. 2. Given the *base* and the *difference* of the hypothenuse and perpendicular of a right angled triangle, to find the perpendicular.

Fig. 10.

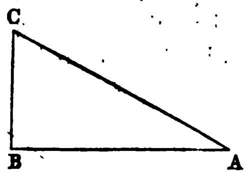
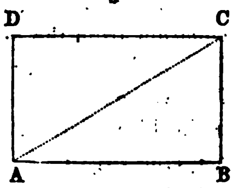


Fig. 11.



Let the base (Fig. 10,)	$AB=b=20$
The perpendicular	$BC=x$
The given difference of AC and BC	$=d=10.$
Then will the hypothenuse	$AC=x+d.$
1. Then by Euclid 47. 1,	$(AC)^2=(AB)^2+(BC)^2$
2. That is, by the notation,	$(x+d)^2=b^2+x^2$
3. Expanding $(x+d)^2$,	$x^2+2dx+d^2=b^2+x^2$
4. Therefore	$x=\frac{b^2-d^2}{2d}=15.$

Prob. 3. If the hypothenuse of a right angled triangle is 30 feet, and the difference of the other two sides 6 feet, what is the length of the base?

Prob. 4. If the hypothenuse of a right angled triangle is 50 rods, and the base is to the perpendicular as 4 to 3, what is the length of the perpendicular?

Prob. 5. Having the perimeter and the diagonal of a parallelogram ABCD, (Fig. 11,) to find the sides.

Let the diagonal $AC=h=10$
The side $AB=x$
Half the perimeter $BC+AB=BC+x=b=14$
Then by transposing x, $BC=b-x$.

Prob. 6. The area of a right angled triangle ABC (Fig. 12,) being given, and the sides of a parallelogram inscribed in it, to find the side BC.

Fig. 12.

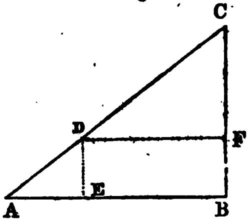
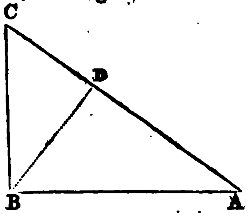


Fig. 13.



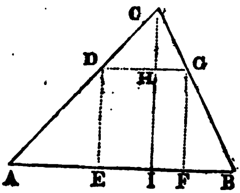
Prob. 7. The three sides of a right angled triangle, ABC, (Fig. 13,) being given, to find the segments made by a perpendicular, drawn from the right angle to the hypothenuse.

The perpendicular will divide the original triangle into two right angled triangles, BCD and ABD. (Euc. 8. 6.)

Prob. 8. Having the area of a parallelogram DEFG (Fig. 14,) inscribed in a given triangle ABC, to find the sides of the parallelogram.

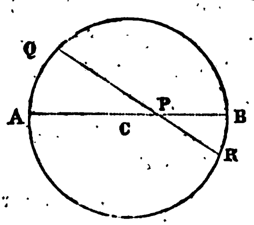
Draw CI perpendicular to AB. By supposition, DG is parallel to AB and C.

Fig. 14.



Prob. 9. Through a given point, in a given circle, so to draw a right line, that its parts, between the point and the periphery, shall have a given difference.

Fig. 15.



In the circle AQBR, (Fig. 15,) let P be a given point, in the diameter AB.

Prob. 10. If the sum of two of the sides of a triangle be 1155, the length of a perpendicular drawn from the angle included between these to the third side be 300, and the difference of the segments made by the perpendicular, be 495; what are the lengths of the three sides?

Prob. 11. If the perimeter of a right angled triangle be 720, and the perpendicular falling from the right angle on the hypothenuse be 144; what are the lengths of the sides?

Prob. 12. The difference between the diagonal of a square and one of its sides being given, to find the length of the sides.

Prob. 13. The base and perpendicular height of any plane triangle being given, to find the side of a square inscribed in the triangle, and standing on the base, in the same manner as the parallelogram DEFG, on the base AB, (Fig. 14.)

Prob. 14. Two sides of a triangle, and a line bisecting the included angle being given; to find the length of the base or third side, upon which the bisecting line falls.

Prob. 15. If the hypothenuse of a right angled triangle be 35, and the side of a square inscribed in it, in the same manner as the parallelogram BEDF, (Fig. 12,) be 12; what are the lengths of the other two sides of the triangle?

Prob. 16. The number of feet in the perimeter of a right angled triangle, is equal to the number of square feet in the area; and the base is to the perpendicular as 4 to 3. Required the length of each of the sides.

Prob. 17. A grass plat 12 rods by 18, is surrounded by a gravel walk of uniform breadth, whose area is equal to that of the grass plat. What is the breadth of the gravel walk?

Prob. 18. The sides of a rectangular field are in the ratio of 6 to 5; and one-sixth of the area is 125 square rods. What are the lengths of the sides?

Prob. 19. There is a right angled triangle, the area of which is to the area of a given parallelogram as 5 to 8. The shorter side of each is 60 rods, and the other side of the triangle adjacent to the right angle, is equal to the diagonal of the parallelogram. Required the area of each.

Prob. 20. There are two rectangular vats, the greater of which contains 20 cubic feet more than the other. Their capacities are in the ratio of 4 to 5; and their bases are squares, a side of each of which is equal to the depth of the other vat. Required the depth of each?

Prob. 21. Given the lengths of three perpendiculars, drawn from a certain point in an equilateral triangle, to the three sides, to find the lengths of the sides.

Prob. 22. A square public green is surrounded by a street of uniform breadth. The side of the square is 3 rods less than 9 times the breadth of the street; and the number of square rods in the street, exceeds the number of rods in the perimeter of the square by 228. What is the area of the square?

Prob. 23. Given the lengths of two lines drawn from the acute angles of a right angled triangle, to the middle of the opposite sides: to find the lengths of the sides.

MISCELLANEOUS PROBLEMS.

1. What two numbers are those whose difference is 10; and if 15 be added to their sum, the amount will be 43? 19 — 9

2. There are two numbers whose difference is 14; and if 9 times the less be subtracted from 6 times the greater, the remainder will be 33. What are the numbers? 31 — 17

3. What number is that to which if 20 be added, and from $\frac{2}{3}$ of this sum, 12 be subtracted, the remainder will be 10? 13

4. What number is that, $\frac{1}{3}$, $\frac{1}{4}$ and $\frac{2}{7}$ of which being added together will make 73? 84.

5. A and B lay out equal sums of money in trade; A gains £120, and B loses £80; and now A's money is triple that of B. What sum had each at first?

6. What number is that, $\frac{1}{3}$ of which exceeds $\frac{1}{4}$ by 72? 864

7. There are two numbers whose sum is 37; and if 3 times the less be subtracted from 4 times the greater, and the remainder be divided by 6, the quotient will be 6. What are the numbers?

8. A man has two children, to $\frac{1}{3}$ of the sum of whose ages if 13 be added, the amount will be 17; but if from half the difference of their ages 1 be subtracted, the remainder will be 2. What is the age of each?

9. A messenger being sent on business, goes at the rate of 6 miles an hour; 8 hours afterwards, another is dispatched with countermanding orders, and goes at the rate of 10 miles an hour. How long will it take the latter to overtake the former?

10. To find two numbers in the proportion of 2 to 3 whose product shall be 54. 9 — 6

11. A man agreed to give a laborer 12*s*. a day for every day he worked, but for every day he was idle he should forfeit 8*s*. After 390 days they settled, and their account was even. How many days did he work?

12. To find a number to the sum of whose digits if 7 be added, the result will be 3 times the left hand digit; and if from the number itself 18 be taken, the digits will be inverted. 53

13. A merchant has two kinds of tea, one of which is worth 9*s*. 6*d*. per pound, the other 13*s*. 6*d*. How many pounds of each must he take to form a chest of 104 lbs. which will be worth £56? 71 - 33

14. To find a number consisting of two digits, the sum of which is 5; and if 9 be added to the number itself, the digits will be inverted. 23

15. There is a certain fraction such, that if you add 1 to its numerator, it becomes $\frac{1}{2}$; but if you add 3 to its denominator, it becomes $\frac{1}{3}$. Required the fraction. $\frac{5}{12}$

16. Out of a cask of water, which had leaked away one third, 21 gallons were drawn, and then being gauged, it was found to be half full. How many gallons did it hold? 126

17. It is required to find two numbers whose difference is 7, and their sum 33. 20 - 13

18. At a town meeting 375 votes were cast, and the person elected to office had a majority of 91. How many votes had each candidate? 142 233

19. A post stands $\frac{1}{4}$ in the ground, $\frac{1}{3}$ in the water, and 10 feet above the water. What is the whole length of it? 24

20. A young man the first day after his arrival at New York, spent $\frac{1}{3}$ of his money, the second day $\frac{1}{4}$, the third day

$\frac{1}{3}$, and he then had only 26 dollars left. How much did he have at first?

21. A person being asked his age, answered that $\frac{3}{4}$ of his age multiplied by $\frac{1}{12}$ of his age, would give a product equal to his age. How many years old was he? 16

22. A man leased a house for 99 years; and being asked how much of the time had expired, replied that two thirds of the time past was equal to four fifths of the time to come. How many years had expired?

23. On commencing the study of his profession, a man found that $\frac{1}{7}$ of his life had been spent before he learned his letters, $\frac{1}{3}$ at a public school, $\frac{1}{3}$ at an academy, and 4 years at college. How old was he?

24. It is required to find a number such, that whether it be divided into two, or three equal parts, the product of its parts will be equal.

25. Two persons, 154 miles apart, set out at the same time to meet each other, one travelling at the rate of 3 miles in 2 hours, the other 5 miles in 4 hours. How long before they meet?

26. A man and his wife usually drank a cask of beer in 12 days, but when the man was absent, it lasted the lady 30 days. How long would it last the man, if his wife were absent?

27. A shepherd being asked how many sheep he had, replied if he had as many more, half as many more, and $7\frac{1}{2}$ sheep, he would then have 500. How many had he?

28. A farmer hired two men to do a job of work for him; one could do the work in 10 days, the other in 15. How long would it take both together to do the same work?

29. A scaffold of hay will keep 5 horses or [illegible] oxen, 87 days. How long will it keep 2 horses and 3 oxen?

30. A and B together can build a boat in 20 days; with the assistance of C, they can do it in 12 days How long would it take C to build the boat?

31. There is a cistern with two aqueducts; one will fill it in 30 minutes, the other will empty it in 40. How long will it take to fill it, if both run together?

32. Required to divide 1 shilling into pence and farthings in such a proportion that there may be 39 pieces.

33. A man divided a small sum of money among his children in the following manner, viz. to the first he gave $\frac{1}{6}$ of the whole $+4$ pence, to the second $\frac{1}{6}$ of the remainder $+8$ pence, to the third $\frac{1}{6}$ of the remainder $+12$ pence, and so on, giving to all an equal sum till he had distributed the whole. Required the number of shares and the sum distributed.

34. A hare has 50 leaps the start of a hound, and takes 4 leaps while the hound takes 3; but 2 leaps of the hound are equal to 3 of the hare. How many leaps will the hound take in catching the hare?

35. A cistern holding 43 gallons, is to be filled in 12 minutes by 2 pipes running alternately. The first runs 4 gallons a minute, and the second 3 gallons a minute. How long did each run?

36. A and B start at the same time and place to go round an island 600 miles in circumference. A goes 30 miles a day, and B 20. How long before they will both be at the starting point together, and how far will each have travelled?

37. A has £100, B £48. A robber takes twice as much from A as from B. A now has 3 times as much as B. What was taken from each?

38. It is required to divide 1200 dols. between A, B and C; B has $256+\frac{1}{2}$ of A's share; C has $270+\frac{1}{2}$ of B's. What was the share of each?

39. There are 3 pieces of cloth of different value. The average price of the first and second is 7 dols. per yard, that of the second and third is 9 dols., and the average price of all is $\frac{5}{8}$ of the third. What are the several prices?

40. A pipe will fill a cistern in 11 hours. After running 5 hours another is opened, and then the two fill it in 2 hours. In what time would the last fill it?

41. A man bought a cask of wine, and $\frac{1}{8}$ of it leaking out, he sold the rest at $2.50 per gallon and neither gained nor lost by his bargain. What did he give per gallon for his wine?

42. A and B start at the same time and in the same direction, but directly opposite each other, to go round a circular pond 536 yards in circumference; A goes 11 yards a minute and B 34 in 3 minutes. In how long time will B overtake A?

43. A pipe fills $\frac{1}{8}$ of a cistern in 1 hour; 2 hours after another is opened and would have hastened the filling of it 1 hour; but 2 hours after, a third begins to discharge, and the cistern is finally filled in the time the first would have filled it. Required the time of the second in filling it, or the third in emptying it?

44. A young man commencing business with a determination to become rich, supported himself for $500 a year, and at the close of every year increased his property by a third part of what remained after his expenses were deducted. In five years he was worth $104,400. What was his original stock?

45. At noon the hour and minute hand of a clock are together. How soon will they be together again?

46. A gentleman bought 5 bushels of wheat and 6 of corn for 27*s.*; he afterwards bought 4 bushels of wheat and 3 of corn for 18*s.* What did he pay per bushel for each?

47. A farmer hired 4 men and 8 boys for a week, and paid them \$40; the next week he hired 7 men and 6 boys for \$50. How much did he pay each by the week?

48. Divide 72 into four such parts that the first increased by 5, the second diminished by 5, the third multiplied by 5, and the fourth divided by 5, the sum, difference, product and quotient, will be equal?

49. A and B can print a certain work in 8 days, A and C in 9 days, B and C in 10 days. How long would it take each one alone to do the work? 14 34/49 17 23/41 23 7/31

50. Three boys were playing marbles; the first game, A loses to B and C, as many as each of them had at the beginning; next, B loses to A and C, as many as each of them had at the end of the first game; last, C loses to A and B, as many as each of them had at the end of the second game. Each then had 16 marbles. How many had each at first? 8 – 26 – 14

51. It is required to find two numbers whose product shall be 54, and the difference of their squares 45. 6 - 9

52. Required the side of a rectangular field which contains 1584 square rods, and its length exceeds its breadth by 8 rods.

53. The united ages of a man and his wife were 42 years, and the product of their ages 432. What was the age of each?

54. A single lady being asked her age, considered the question impertinent and gave an evasive answer, saying; "if you take 4 years from my age, and extract the square root of the remainder, and multiply the root by 4, and add 4 to the product, the sum will be 24." What was her age? 29

55. A peach orchard which contained 900 trees was so planted, that there were 11 rows more than there were trees

in a row. How many rows were there, and how many trees in each row? 36 – 75

56. A man purchased a tract of land in a square form, which contained as many acres as there were rails in the fence by which it was enclosed; the rails were 11 feet long and the fence was 4 rails high. How many acres did the tract contain?

57. A man bought 80 pounds of pepper and 36 pounds of saffron, so that for 8 crowns he had 14 pounds of pepper more than of saffron for 26 crowns; and the amount he laid out was 188 crowns. How many pounds of pepper did he buy for 8 crowns?

58. It is required to find four numbers in arithmetical progression such, that the product of the extremes shall be 45, and the product of the means 77. 3 – 7 – 11 – 15

59. It is required to find three numbers in geometrical progression such, that their sum shall be 14, and the sum of their squares 84. 2 – 4 – 8

60. The hypothenuse of a right angled triangle is 13 feet, and the difference between the other two sides is 7. Required the sides. 5 – 12

61. The perpendicular of a plane triangle is 300 feet; the sum of two of the sides is 1150 feet, and the difference of the segments of the base is 495 feet. Required the base and the sides.

62. In a plane triangle the base is 50 feet, the area 796 feet, and the difference of the sides is 10 feet. Required the sides and perpendicular.

THE END.

In the hurry of bringing out this edition some errors, almost as a matter of course, have crept in. Care will be taken to have the future editions accurate. A few hundred copies have gone out without these corrections.

ERRATA.

Page 44, Art. 91, Ex. 2, for $2a$ dollars, read $3a$ dollars.

" 68, " 134, 4 lines from bottom, for $\frac{d}{h \times y}$ read $\frac{d}{h+y}$.

" 93, " 163, line 7, for $\frac{1}{d^2}=d^2$, read $\frac{1}{d^2}=d^{-2}$.

" 110, " 192, line 7, for $a^5 \div a$, read $a^5 \div a^3$.

" 138, " 239, Ex. 17, for $(r^2y)^{3\frac{3}{7}}$, read $(r^2y^3)^{\frac{3}{7}}$.

" 156, " 263, Ex. 17, for *first number*, read *first member*.

" 173, " 277, Case I, for *one or two*, &c. read *one of two*.

" 193, " 308, line 7 from bottom, for *one compound of*, read *one compounded of*, &c.

" 223, " 365, lines 5 and 6, for $a+r$ and ar^2+r, read $a \times r$, &c.